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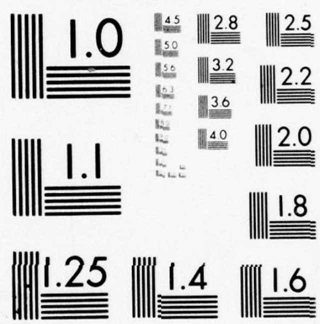

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QUEUEING NETWORKS IN HEAVY TRAFFIC

BY

MARTIN IRA REIMAN

TECHNICAL REPORT NO. 76

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CHAPTER 1

INTRODUCTION AND SUMMARY

The principal purpose of this dissertation is to state and prove a limit theorem which justifies a diffusion approximation for general queueing networks. In this chapter we begin by presenting a known result for single server queues in heavy traffic. The main result of this dissertation is the extension of this single server queue result to general queueing networks. A description of the class of queueing networks considered is given in Section 2. In the same section we also describe the associated process of interest to us here, $\underset{\sim}{Q} = \{\underset{\sim}{Q}(t),\ t \geq 0\}$, the vector process of queue lengths in the network. We then describe exact results which have been previously obtained for such queueing networks. The next three sections of this chapter briefly describe the new results of this dissertation. Section 4 contains a definition of the vector diffusion process $\underset{\sim}{Z}^* = \{\underset{\sim}{Z}^*(t),\ t \geq 0\}$ which is advanced as an approximation for $\underset{\sim}{Q}$. The limit theorem justifying the approximation is stated in Section 5, and some properties of the process $\underset{\sim}{Z}^*$ are discussed in Section 6. In Section 7 we show how one chooses the correct diffusion approximation for a particular queueing network. Finally, we present some of the notational conventions used in this dissertation in Section 8.

In Chapter 2 we present technical preliminaries from weak convergence theory. We construct the probability space which supports our queueing network in Chapter 3. In addition, we construct a random walk

and reflected random walk on this probability space, and prove appropriate weak convergence limit theorems for these processes. In Chapter 4 we construct our queueing network on the probability space introduced in Chapter 3. We also construct two modified queueing networks, which are used in the proof of the main limit theorem. The main result of this dissertation, a weak convergence limit theorem for the vector queue length process, is stated and proven in Chapter 5. We study properties of the limit process in Chapter 6. Finally, in Chapter 7 we indicate how our limit theorem can be extended to cover more general types of queueing networks, such as those with multi-server stations.

1.1. A Heavy Traffic Limit Theorem for Single Server Queues

In this section we state a known result for GI/G/1 queues in heavy traffic. This result is actually a special case of a theorem for more general systems treated by Iglehart and Whitt (1970a,b). These results, as well as other heavy traffic work done before 1973, are surveyed by Whitt (1974).

We begin with the standard definition of a GI/G/1 queue. Thus we have mutually independent sequences of i.i.d. random variables which function as interarrival and service times for the system. The interarrival times are distributed as u and the service times are distributed as v. We define

$$\mu = 1/E[v] > 0 , \qquad s = var(v) > 0,$$
$$\lambda = 1/E[u] > 0, \qquad a = var(u) > 0 ,$$

$$\sigma^2 = \lambda^3 a + \mu^3 s,$$

$$\rho = \lambda/\mu, \quad \text{and} \quad \alpha = 1-\rho .$$

The queue length process $Q = \{Q(t),\ t \geq 0\}$ is constructed in the standard manner using the interarrival and service time sequences.

Roughly speaking, we say that the GI/G/1 system is in heavy traffic when the traffic intensity ρ is close to unity. The precise mathematical interpretation of heavy traffic involves consideration of a sequence of queueing systems, indexed (say) by $n = 1,2,\ldots,$ each with its own parameters and defined in such a way that $\rho_n \to 1$. We denote sequential dependencies by appending indices on quantities of interest. We assume that $\alpha_n \downarrow 0$ as $n \to \infty$ meaning that $\rho_n \to 1$ from below as $n \to \infty$. Before stating the limit theorem we need to define the limit process, which in this one-dimensional case is reflected Brownian motion. Let $X = \{X(t),\ t \geq 0\}$ be a Brownian motion with drift c and variance σ^2. Define

$$Z(t) = X(t) + Y(t) , \qquad t \geq 0 ,$$

where

$$Y(t) = - \inf_{0 \leq s \leq t} \{X(s)\} .$$

Then $Z = \{Z(t),\ t \geq 0\}$ is reflected Brownian motion. Note that $Y = \{Y(t),\ t \geq 0\}$ could alternately be defined as the minimal nondecreasing and nonnegative function of X such that $Z(t) \geq 0$ for all $t \geq 0$.

THEOREM. Suppose

i) $\alpha_n \downarrow 0$ as $n \to \infty$,

ii) $a(n) \to a$ as $n \to \infty$,

iii) $s(n) \to s$ as $n \to \infty$,

iv) $[\lambda(n) - \mu(n)]/\alpha_n \to c$ as $n \to \infty$,

v) $\sup_{n \geq 1} E([v^1(n)]^{2+\epsilon}) < \infty$ for some $\epsilon > 0$,

vi) $\sup_{n \geq 1} E([u^1(n)]^{2+\epsilon}) < \infty$ for some $\epsilon > 0$.

Then $\alpha_n Q^n(\cdot/\alpha_n^2) \Rightarrow Z(\cdot)$ in D as $n \to \infty$, where $Z = \{Z(t), t \geq 0\}$ is reflected Brownian motion with drift c and variance $\sigma^2 = \lambda^3 a + \mu^3 s$.

The interpretation of the above theorem which is useful for applications is that for small α, $\alpha Q(t/\alpha^2)$ is distributed approximately like $Z(t)$.

1.2. The General Network Model

The queueing network model considered here is identical to that first studied by Jackson (1957), with two changes. First, we only consider a network with single server stations. (This is only for ease of exposition. The multi-server case is covered in Chapter 7.) The second and more important change is that Jackson's distributional assumptions have been dropped. He assumed exponential interarrival and service times, but we allow a general form for these distributions.

The network consists of K stations, each of which acts as a single server queue. An example with 2 stations is pictured schematically in Figure 1. There is no bound placed on the queue length at any station.

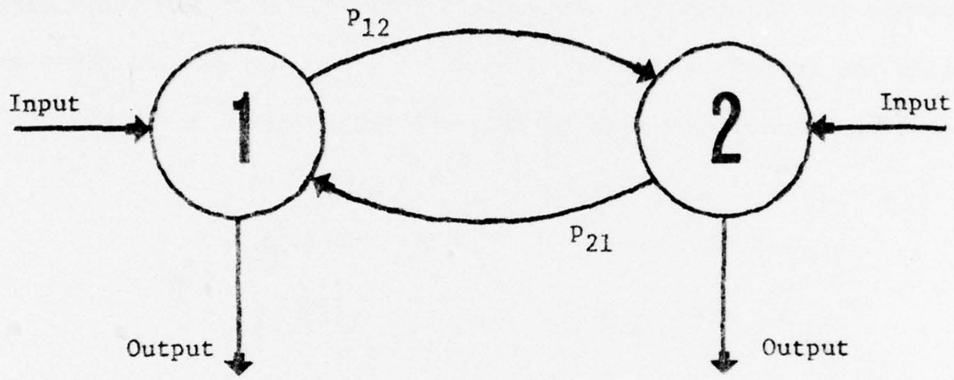

FIGURE 1. The Two Station Network

For the sake of concreteness, we assume the queue discipline to be first-come-first-served. An easy generalization of this is given in Chapter 7. Each station may have its own input process from outside the network, and at least one station must have such an input process. In the conventional terminology, we are dealing here with an open network. Customers completing service at station i are routed to station j (where they join the queue) with probability p_{ij}. Thus the routing is described by a $K \times K$ matrix P, which must be substochastic in the sense that at least one row sum must be strictly less than one

and no square submatrix can be stochastic. The defect of each row gives the probability for a customer to leave the network after completing service at the corresponding station.

We assume an underlying probability space $(\Omega,\mathcal{F},P)$ on which are defined mutually independent sequences, of i.i.d. random variables which are used as interarrival times, service times and routing indicators for the network. The interarrival times to station k (from outside the system) are distributed as u^k, and the service times at station k are distributed as v^k. We define (for $1 \leq k \leq K$)

$$\mu_k = 1/E[v^k] > 0 , \qquad s_k = \mathrm{var}(v^k) > 0 ,$$

$$\lambda_k = 1/E[u^k] \geq 0 , \qquad a_k = \mathrm{var}(u^k) \geq 0 .$$

Denote by $A_k(t)$ the number of customers who have arrived to station k from outside the system during the interval $[0,t]$. Thus $A_k(t) \equiv 0$ for $t \geq 0$ if station k has no input process from outside the system. Our assumption of mutually independent i.i.d. interarrival time sequences is equivalent to assuming that the A_k are independent renewal processes.

Associated with station k we define a process $\{S_k(t), t \geq 0\}$ where $S_k(t)$ denotes the number of customers who have completed service at station k during the interval $[0,t]$. Additionally, let $S_{ij}(t)$ denote the number of customers who have completed service at station i during the interval $[0,t]$ and were then immediately routed to station j. If $p_{ij} = 0$ then $S_{ij}(t) \equiv 0$ for all $t \geq 0$. We clearly have

$$S_k(t) \geq \sum_{j=1}^{K} S_{kj}(t) .$$

We will henceforth assume that $p_{ii} = 0$, so that $S_{ii}(t) \equiv 0$ for all $t \geq 0$. This assumption is without loss of generality. (See Chapter 7.)

The process of interest to us here is the vector process of queue lengths (including customers in service, if any) at the stations of the network. Let $Q_k(t)$ denote the number of customers at station k (called the queue length from now on) at time t. We then obviously have

$$Q_k(t) = A_k(t) + \sum_{j=1}^{K} S_{jk}(t) - S_k(t). \tag{2}$$

Let $\underset{\sim}{Q}(t) = (Q_1(t), Q_2(t), \ldots , Q_K(t))$. To complete the description of the vector queue length process we need to specify initial conditions. We assume that the entire network is empty at $t = 0-$. This particular initial condition makes the statement and proof of the limit theorem somewhat neater than it would otherwise be. The initial conditions actually have no effect on limit theorems like ours, as has been pointed out by Whitt (1968).

We now define a traffic intensity for each station in the network, following Jackson (1957). The traffic intensities are defined in terms of quantities γ_k, $1 \leq k \leq K$, which we call 'net arrival rates'. We define $\underset{\sim}{\gamma} = (\gamma_1, \ldots, \gamma_K)$ as the unique solution of the equation

$$\underset{\sim}{\gamma} = \underset{\sim}{\lambda} + \underset{\sim}{\gamma}P \tag{3}$$

where $\underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda_K)$ is the vector of arrival rates and P is the routing matrix. Since P is strictly substochastic it is well known (cf. Çinlar (1975)) that $(I-P)$ is invertible and

$$R \equiv (I-P)^{-1} = \sum_{n=0} P^n .$$

From the definition of P it is then easy to show that R_{ij} represents the expected number of visits to station j by a customer who enters the network at station i. Clearly (3) is uniquely solved by $\underset{\sim}{\gamma} = R\underset{\sim}{\lambda}$ and the interpretation of γ_j as a net arrival rate at station j then follows from the interpretation of R. The traffic intensity at station j can now be defined as

$$\rho_j = \frac{\gamma_j}{\mu_j} , \qquad 1 \leq j \leq K .$$

1.3. Literature of Queueing Networks

The first work on queueing networks was done over twenty years ago by Jackson (1957). The network model he considered, as explained in Section 1, is the same as that described above--with two changes. He allowed an arbitrary finite number of servers at each station and required that all interarrival and service time distributions be exponential.

Jackson studied the vector stochastic process representing the number of customers at each station in the network. Because of his

distributional assumptions this process is Markov. Jackson defined the traffic intensities ρ_i, $1 \leq i \leq K$, as in the last section. Let $\pi(n_1,\ldots,n_K)$ be the stationary probability of state $(n_1,\ldots,n_K)$ when it exists.

Jackson's main result (specialized to the case of a single server at each station) states that the stationary distribution exists if $\rho_i < 1$ for all $i = 1,\ldots,K$ and

$$\pi(n_1,\ldots,n_K) = \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i} \quad (1)$$

Since Jackson's original paper, a great deal of work has been done aimed at extending his results. The main objective of this work has been to increase the complexity of the network while retaining the product form solution for the stationary distribution. We will be rather brief in our discussion since this literature has been surveyed recently by Lemoine (1977).

The first extension was actually made by Jackson (1963) himself. He extended his original model to allow the arrival processes to depend on the number of customers in the network. He also allowed the service rate at a station to depend on the number of customers at that station. Gordon and Newell (1967) treated a closed network (fixed total number of customers). They allowed the same service rate dependence as in Jackson (1963). Posner and Bernholtz (1967) generalized the results of Gordon and Newell. They allowed several different classes of customers in the network, each class having its own routing probabilities and service rates. They also allowed arbitrarily distributed travel times between stations. Kelly (1975) considered both open and closed

networks with different classes of customers and different types of service. The service stations could be multiserver, infinite server or processor sharing.

In all of the work described above the one common restriction is to exponentially distributed service times. There has been some work done aimed at lifting this restriction. Perhaps the most general model is that of Baskett, Chandy, Muntz and Palacios (1975) (further generalized by Kobayashi and Reiser (1975)). They allow several classes of customers, with customers able to change class upon leaving a station. The service distribution, routing probabilities and arrival processes all can depend on the class. All external arrival processes are Poisson processes. It is possible for the network to be open with respect to some classes of customers and closed with respect to the others. The service stations may be one of four types: first-come-first-served (FCFS), processor sharing, infinite server, and last-come-first-served pre-emptive resume. The service times at all but FCFS stations may have any distribution with rational Laplace transform. At FCFS stations the service time distribution must be exponential and all classes must have the same distribution.

Kelly (1976) deals with a system having several classes of customers where each class has a fixed route through the network. Kelly gets results similar to those of Baskett et al., showing that for all types of disciplines except FCFS a product form solution is obtained when service times are finite mixtures of Gamma distributions. Barbour (1976) extends this result to hold for all arbitrarily distributed service times using weak convergence arguments.

Although some of the recent work described above has dealt with models having non-exponential service times these less restrictive distributions have been tied to disciplines other than first-come-first-served. A closer examination of these results will show that the stationary distributions, all roughly of the form (1), only depend on the means of the service times. Thus the variance (and higher moments) of service times, which are known to have a significant impact in single server queues, have effectively been ignored in all the work described above.

There has also been work of a different character done on queueing networks. This involves analyzing networks by decomposing them into smaller pieces, analyzing the pieces and then putting them back together. This research has thus far been aimed towards characterizing output processes from single server queues and studying the behavior of queues which use these output processes as input processes. Work has also been done on feedback queues as well as thinning and superimposing output flows. Much progress has been made but no results have been obtained which enable a discussion of queue length distributions in a network. Work in this area is dealt with in a survey article by Disney (1975).

1.4. The Approximating Diffusion for a Network

Consider a K dimensional Brownian motion $\underset{\sim}{X}^* = \{\underset{\sim}{X}^*(t), t \geq 0\}$ with drift vector $\underset{\sim}{c}$ and covariance matrix Σ. Define

(2) $$\underset{\sim}{Z}^*(t) = \underset{\sim}{X}^*(t) + (I-P^T)\,\underset{\sim}{Y}(t)$$

where P is the routing matrix and $\underset{\sim}{Y} = \{\underset{\sim}{Y}(t);\ t \geq 0\}$ is described as the minimal (component by component) non-decreasing function of $\underset{\sim}{X}^*$ such that $\underset{\sim}{Z}^*(t) \geq 0$ for all $t \geq 0$. It will be shown in Section 3.2 that this definition of $\underset{\sim}{Y}$ and hence $\underset{\sim}{Z}^*$ is in fact unambiguous.

The process $\underset{\sim}{Z}^*$ is a diffusion whose state space is the non-negative orthant of $\mathbb{R}^K$, hereafter denoted $\mathbb{R}^K_+$. On the interior of its state space, $\underset{\sim}{Z}^*$ behaves like the Brownian motion $\underset{\sim}{X}^*$. It reflects in the direction $e_i - p_{i.}$ from the boundary corresponding to $Z_i^*(t) = 0$, where e_i is a unit vector in direction i, and $p_{i.}$ is the ith row of P). For the network depicted in Figure 1, the directions of reflection are shown in Figure 2.

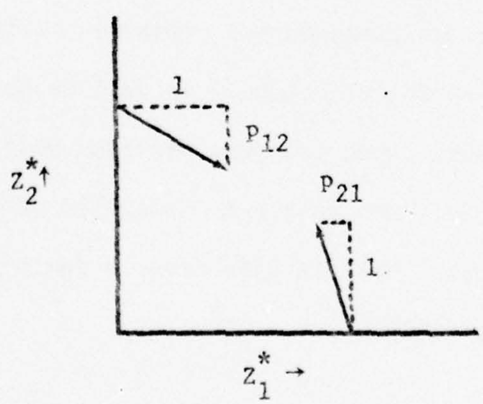

FIGURE 2. Directions of Reflection for a 2 Station Network

1.5. The Limit Theorem

We now consider a sequence of queueing networks of the type defined in Section 2, indexed by $n \geq 1$. The notation of Section 2 is retained, with indices appended to quantities of interest. Heavy traffic for the network means that $\rho_i^n \to 1$ as $n \to \infty$ for $i = 1,\ldots,K$. It is convenient to describe the result in terms of $\alpha_n = \min_{1 \leq i \leq K} (1-\rho_i^n)$. In this section we deal only with the case where $\rho_i^n \to 1$ from below for all $i = 1,\ldots,K$.

THEOREM. Suppose

i) $\alpha_n \downarrow 0$ as $n \to \infty$,

ii) $a_i(n) \to a_i$ as $n \to \infty$ for $1 \leq i \leq K$,

iii) $s_i(n) \to s_i$ as $n \to \infty$ for $1 \leq i \leq K$,

iv) $[\nu_i(n) - \mu_i(n)]/\alpha_n \to c_i$ as $n \to \infty$ for $1 \leq i \leq K$,

v) $\max_{1 \leq i \leq K} \sup_{n \geq 1} E([v_i^1(n)]^{2+\epsilon}) < \infty$ for some $\epsilon > 0$,

vi) $\max_{1 \leq i \leq K} \sup_{n \geq 1} E([u_i^1(n)]^{2+\epsilon}) < \infty$ for some $\epsilon > 0$.

Then $\alpha_n \underset{\sim}{Q}^n(t/\alpha_n^2) \Rightarrow \underset{\sim}{Z}^*(t)$ as $n \to \infty$ where $\underset{\sim}{Z}^* = \{\underset{\sim}{Z}^*(t),\ t \geq 0\}$ is Brownian motion in $\mathbb{R}_+^K$ with drift vector $\underset{\sim}{c}$, covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} \sigma^2_{11} & \sigma^2_{12} & \cdot & \cdot & \cdot & \sigma^2_{1K} \\ \sigma^2_{12} & \sigma^2_{22} & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & \cdot & \\ \sigma^2_{1K} & & & & & \sigma^2_{KK} \end{pmatrix},$$

with

(3) $$\sigma^2_{ii} = \lambda^3_i a_i + \mu^3_i s_i + \sum_{j=1}^{k} \mu_\ell p_{\ell k}(1 - p_{\ell k}\mu^2_\ell s_\ell) \quad \text{for} \quad 1 \leq i \leq K$$

and

(4) $$\sigma^2_{ij} = -[\mu^3_i s_i p_{ij} + \mu^3_j s_j p_{ji} + \sum_{\ell=1}^{K} \mu_\ell p_{\ell i} p_{\ell j}(\mu^2_\ell s_\ell - 1)] \text{ for } 1 \leq i \leq j \leq K,$$

and reflection in direction $e_i - p_{i.}$ from the boundary corresponding to $Z^*_i = 0$.

In Chapter 5 the limit theorem is stated and proven. There we take $\alpha_n = n^{-1/2}$ with no loss of generality. This choice for α_n makes it easier to refer to existing results from weak convergence theory.

1.6. Properties of the Diffusion Limit

Working with known results from the theory of Markov processes, a partial differential equation for the stationary distribution of $\underset{\sim}{Z}^*$ is derived in Chapter 6. It is shown there that in order for a probability density function π to be the stationary distribution of $\underset{\sim}{Z}^*$ it is necessary that it satisfy

$$(5)\qquad \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sigma_{ij}^2 \frac{\partial^2 \pi}{\partial x_i \partial x_j} - \sum_{i=1}^{K} c_i \frac{\partial \pi}{\partial x_i} = 0$$

$$(6)\qquad \frac{1}{2} \sigma_i^2 \frac{\partial \pi}{\partial x_i} + \sum_{j \neq i} \left(\frac{1}{2} \sigma_i^2 p_{ij} + \sigma_{ij}^2\right) \frac{\partial \pi}{\partial x_j} - c_i \pi = 0$$

$$\text{for } x_i = 0,\ x_j > 0,\ j \neq i,\ 1 \leq i \leq K$$

(where $\sigma_i^2 \equiv \sigma_{ii}^2$, $1 \leq i \leq K$).

These equations have not been solved in general, but for the family of networks corresponding to

$$\sigma_{ij}^2 = -\frac{1}{2} (\sigma_i^2 p_{ij} + \sigma_j^2 p_{ji}), \qquad i \neq j$$

$$\frac{-(ck)_k}{\sigma_k^2} = d_k > 0 \qquad 1 \leq k \leq K$$

we have

$$(7)\qquad \pi(x_1, \ldots, x_K) = \prod_{k=1}^{K} d_k e^{-d_k x_k} .$$

As might be suspected from the form of the stationary distribution, this family of limit processes corresponds precisely to limits achievable by sequences of Jackson networks.

1.7. Approximating Stable Queueing Networks

The limit theorem stated in Section 5 suggests that we approximate $\alpha \underset{\sim}{Q}(t/\alpha^2)$ by $\underset{\sim}{Z}^*(t)$ when α is small. The parameters of the limit process $\underset{\sim}{Z}^*$ arise as limits of parameters from a sequence of queueing networks. In general, one has a single network for which a diffusion approximation is desired. The correct choice of parameters for $\underset{\sim}{Z}^*$ is straightforward. It involves matching the infinitesimal drifts and covariances of $\underset{\sim}{Q}$ and $\underset{\sim}{Z}^*$. Carrying out these calculations, we obtain the following values for the parameters of $\underset{\sim}{Z}^*$:

$$\alpha = \min_{1 \leq i \leq K} (1-\rho_i) > 0 ,$$

$$c_i = (\nu_i - \mu_i)/\alpha \qquad \text{for } 1 \leq i \leq K,$$

$$\sigma_{ii}^2 = \lambda_i^3 a_i + \mu_i^3 s_i + \sum_{\ell=1}^{K} \mu_\ell^3 s_\ell p_{\ell i}^2 \qquad \text{for } 1 \leq i \leq K,$$

and

$$\sigma_{ij}^2 = -[\mu_i^3 s_i p_{ij} + \mu_j^3 s_j p_{ji} + \sum_{\ell=1}^{K} \mu_\ell p_{\ell i} p_{\ell j} (\mu_\ell^2 s_\ell - 1)] \qquad \text{for } 1 \leq i < j \leq K.$$

The reflection directions are given by $\underset{\sim}{e}_i - \underset{\sim}{p}_{i.}$ for $1 \leq i \leq K$.

Although we offer no rigorous justification, it is tempting to approximate the stationary distribution of $\alpha \underset{\sim}{Q}$ by that of $\underset{\sim}{Z}^*$. This is the reason for our interest in the stationary distribution of $\underset{\sim}{Z}^*$.

The use of a diffusion approximation for both transient and stationary distributions of queueing networks was originally proposed by Kobayashi (1974). The open network that he studied is essentially equivalent to that treated here. The diffusion approximation he advanced is a multidimensional Brownian motion restricted to the non-negative orthant. His work involves no proofs, but contains a heuristic justification. He never explicitly considered the behavior of the diffusion at the boundaries, which is more complicated than in the one-dimensional case. As a result of this, the diffusion was never completely specified. Kobayashi wrote out the forward equation for the Brownian motion and dealt with the related steady-state solution, obtained as the solution to the forward equation with the time derivative set to zero. The equation he solved was precisely (5). Although Kobayashi obtained a solution to (5), it does not always satisfy (6). There are, however, certain cases for which his solution is correct, and these are discussed in Chapter 6.

1.8. Notation

We briefly indicate here some of the notational conventions used in this dissertation. Other notation will be explained when it is introduced. Propositions, lemmas, and equations are numbered consecutively

within chapters, with the numbering starting anew in each chapter. Within the same chapter, propositions, lemmas, and equations are referenced simply by their number, whereas cross chapter references also require a chapter number. Thus equation (proposition, lemma) 4 of Chapter 2 is referenced to as (2.4) outside of Chapter 2, and as (4) inside Chapter 2. Theorems, on the other hand, are numbered consecutively throughout the dissertation and thus require only one number for identification.

When there is any chance of confusion, we underscore vectors with ~. All vector functions and vector stochastic processes are underscored with ~. Matrices will be obvious from context and so are not indicated in any special way. We use several different norms, most of which we denote by $|\cdot|$. The specific norm used is generally obvious from the nature of its argument. When there is a chance of confusion, the type of norm being used is explicitly stated.

CHAPTER 2

PRELIMINARIES

In this chapter we present results from weak convergence theory which will be used in the proof of the main limit theorem. The first section contains known results and the second section contains two new results.

2.1. Weak Convergence

We begin by briefly sketching the notation and terminology that we use for weak convergence. The reader is assumed to have a basic familiarity with this material. Billingsley (1968) is the standard reference in this area.

We will consider a sequence of processes with paths in $D^K[0,1]$ ($\equiv D^K$), the space of all right continuous functions from $[0,1]$ into $\mathbb{R}^K$ having left limits. This space is a complete separable metric when endowed with the Skorohod metric d. We never need to explicitly deal with d and so we do not define it here. Our limit process has paths in $C^K[0,1]$ ($\equiv C^K$), the space of all continuous functions from $[0,1]$ into $\mathbb{R}^K$. The metric for C^K is

$$\rho(x,y) = \sup_{a \leq t \leq 1} |x(t) - y(t)| ,$$

for $x, y \in C^K$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^K$. Under the metric ρ, C^K is a complete, separable metric space.

We use the symbol $\Rightarrow$ to denote weak convergence. The same symbol is used for both probability measures and random functions.

Although the sequence of stochastic processes we deal with are in D^K, we can effectively treat them as elements of C^K when it is convenient. This is because of the following result of Liggett and Rosen, proven in Whitt (1968).

Proposition 2. Let $\{X_n, n \geq 1\}$ be a sequence of random functions in (D,d), $\{Y_n, n \geq 1\}$ a sequence of random functions in (C,ρ), and X a random function in (C,ρ). If $d(X_n,Y_n) \Rightarrow 0$ as $n \to \infty$, then $X_n \Rightarrow X$ in (D,d) as $n \to \infty$ if and only if $Y_n \Rightarrow X$ in (C,ρ) as $n \to \infty$.

The generalization of the result to C^K and D^K is immediate. We will never need to deal with d because $d(x,y) \leq \rho(x,y)$ for $x, y \in D^K$.

We now state several results for weak convergence of stochastic processes formed from i.i.d. random variables. The limit process is Brownian motion in all cases. The first result is due to Prohorov (1956) and is stated here under slightly stronger conditions than are necessary, since they are easier to verify in our applications.

Proposition 2. Let $\{X_i^n, i \geq 1, n \geq 1\}$ be a double sequence of random variables which are i.i.d. for each $n \geq 1$. Assume that

i) $E[X_1^n] = 0$, $n \geq 1$,

ii) $Var(X_1^n) = \sigma_n^2 > 0$, $n \geq 1$,

iii) $\sigma_n^2 \to \sigma^2$ as $n \to \infty$, $\sigma^2 > 0$,

and iv) $\sup_{n \geq 1} E\{|X_1^n|^{2+\epsilon}\} < \infty$ for some $\epsilon > 0$.

Let

$$\beta^n(t) = (\sigma^2 n)^{-1/2} \sum_{j=1}^{[nt]} X_j^n \quad \text{for } 0 \leq t \leq 1 \text{ and } n \geq 1,$$

where $[\cdot]$ denotes the integer part of the argument. Then $\beta^n \Rightarrow \beta$ in (D,d) where β is standard (no drift, unit variance) Brownian motion.

The following is an associated result for a process formed from a random sum of random variables. It is proved as Theorem 17.1 of Billingsley (1968).

Proposition 3. Let $\{X_i^n, i \geq 1, n \geq 1\}$ be a double sequence of random variables satisfying the hypotheses of Proposition 2. Let $\{N(t), t \geq 0\}$ be a renewal process with rate λ $(0 < \lambda < \infty)$, and let

$$\gamma^n(t) = (\sigma^2 \lambda n)^{-1/2} \sum_{j=1}^{N(nt)} X_j^n \quad \text{for } 0 \leq t \leq 1 \text{ and } n \geq 1.$$

Then $\gamma^n \Rightarrow \beta$ in (D,d).

The next result deals with weak convergence of renewal processes. It is Theorem 17.3 of Billingsley (1968).

Proposition 4. Let $\{u_i^n, i \geq 1, n \geq 1\}$ be a double sequence of positive random variables which are i.i.d. for each $n \geq 1$. Assume that $E[u_1^n] = \lambda^n$ $(0 < \lambda^n < \infty)$ and $\lambda^n \to \lambda$ $(0 < \lambda < \infty)$ as $n \to \infty$. Let $X_i^n = u_i^n - \lambda^n$ and assume that $\{X_i^n, i \geq 1, n \geq 1\}$ satisfies the hypotheses of Proposition 2. For $n \geq 1$ let

$$A^n(t) = \begin{cases} \max\{j \geq 1 \mid \sum_{i=1}^{j} u_i^n \leq t\} & \text{for } u_1^n \leq t \\ 0 & \text{for } u_1^n > t \end{cases}$$

and let

$$\eta^n(t) = (\sigma^2\lambda^3 n)^{-1/2}\,[A^n(nt) - \lambda^n nt] \quad \text{for } 0 \leq t \leq 1 \text{ and } n \geq 1.$$

Then $\eta^n \Rightarrow B$ in (D,d) as $n \to \infty$.

2.2. Asymptotic Nonpositivity of Supermartingales

We now present two results dealing with supermartingales which show that in a certain sense they are asymptotically nonpositive. These results appear to be new and so are proven here.

Let $(\Omega,\mathcal{F},P)$ be a probability space with an increasing sequence of sub σ-fields $\{\mathcal{F}_n, n \geq 0\}$, so that $\mathcal{F}_n \subset \mathcal{F}$ for all $n \geq 0$. Let $\{x_n, n \geq 1\}$ be a sequence of random variables such that x_n is $\mathcal{F}_n$ measurable for all $n \geq 1$. Assume that

$$E[x_n | \mathcal{F}_{n-1}] \le 0 \quad \text{for} \quad n \ge 1$$

and

$$\operatorname{var}[x_n | \mathcal{F}_{n-1}] \le C < \infty \quad \text{for} \quad n \ge 1.$$

If we set

$$X_n = \sum_{i=1}^{n} x_i ,$$

then $\{X_n, n \ge 1\}$ is a supermartingale adapted to $\{\mathcal{F}_n, n \ge 0\}$. Let

$$y_i = x_i - E[x_i | \mathcal{F}_{i-1}] \quad \text{for} \quad i \ge 1$$

and

$$Y_n = \sum_{i=1}^{n} y_i \qquad \text{for} \quad n \ge 1.$$

Then $\{Y_n, n \ge 1\}$ is a martingale adapted to $\{\mathcal{F}_n, n \ge 0\}$ with $Y_n \ge X_n$ for $n \ge 1$.

Proposition 5. Suppose that $\{\beta^n, n \ge 1\}$ is a sequence of random functions in (D,d) with $\beta^n \Rightarrow \beta$ in (D,d), where β is a Brownian motion with finite drift and variance. Let

$$z^n = - \inf_{0 \le t \le 1} \{\beta^n(t)\} .$$

Then

$$P\{ \max_{1 \le \ell \le z^n \sqrt{n}} X_\ell > \sqrt{n}\, \epsilon\} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \epsilon > 0$$

and

$$P\{ \max_{1 \le \ell \le z^n \sqrt{n}} |Y_\ell| > \sqrt{n}\, \epsilon\} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \epsilon > 0.$$

Proof. Let $z = - \inf_{0 \le t \le 1} \{\beta(t)\}$. The continuous mapping theorem (Billingsley (1968), Theorem 5.5) gives $z^n \Rightarrow z$ in R since $\beta^n \Rightarrow \beta$ in D. By well known results for Brownian motion, z has finite expectation and hence is almost surely finite. Thus for every $\delta > 0$, there exists an $\eta < \infty$ such that

$$P\{z \le \eta\} \ge 1 - \frac{\delta}{4} .$$

In addition, there exists an $N < \infty$ such that if $n \ge N$, then

$$|P\{z \le \eta\} - P\{z^n \le \eta\}| \le \frac{\delta}{4} ,$$

because $z^n \Rightarrow z$. Putting these two facts together,

$$P\{z^n \le \eta\} \ge 1 - \frac{\delta}{2} \quad \text{for} \quad n \ge N .$$

We work henceforth on the set $\{z^n \le \eta\}$ where

$$\max_{1 \le \ell \le z^n \sqrt{n}} X_\ell \le \max_{1 \le \ell \le \eta \sqrt{n}} X_\ell$$

and

$$\max_{1 \le \ell \le z^n \sqrt{n}} |Y_\ell| \le \max_{1 \le \ell \le \eta \sqrt{n}} |Y_\ell| .$$

By the maximal inequality for martingales (Chung (1974, Theorem 9.4.1))

$$P\{\max_{1 \le \ell \le \eta \sqrt{n}} |Y_\ell| > \sqrt{n}\,\epsilon\} \le (n\epsilon^2)^{-1} \operatorname{var}(Y_{[\eta \sqrt{n}]}) .$$

For $1 \le i < j$, $E[Y_i Y_j] = E(E[Y_i Y_j | \mathcal{F}_{j-1}]) = E(Y_i E[Y_j | \mathcal{F}_{j-1}]) = 0$ by the independence of Y_i and $E[Y_i | \mathcal{F}_{j-1}]$. Thus $\mathrm{var}(Y_{[\eta\sqrt{n}]}) \le C\eta\sqrt{n}$. Choose $N' > N$ so that

$$C\eta/\epsilon^2 \sqrt{N'} < \frac{\delta}{2}$$

and the result for Y_ℓ follows. The result for X_ℓ is obtained upon noting that

$$X_\ell \le Y_\ell \le |Y_\ell| .$$

Proposition 6. If $E[x_i | \mathcal{F}_{i-1}] \le -q < 0$ for $i \ge 1$ then

$$P\{\max_{1 \le \ell \le n} X_\ell > \sqrt{n}\,\epsilon\} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \epsilon > 0 .$$

Proof. We have

$$\begin{aligned} P\{\max_{1\le\ell\le n} X_\ell > \sqrt{n}\,\epsilon\} &= P\{\max_{1\le\ell\le\sqrt{n}} Y_\ell > \sqrt{n}\,\epsilon/2,\ \max_{1\le\ell\le n} X_\ell > \sqrt{n}\,\epsilon\} \\ &\quad + P\{\max_{1\le\ell\le\sqrt{n}} Y_\ell \le \sqrt{n}\,\epsilon/2,\ \max_{\sqrt{n}\le\ell\le n} X_\ell > \sqrt{n}\,\epsilon\} \\ &\le P\{\max_{1\le\ell\le\sqrt{n}} Y_\ell > \sqrt{n}\,\epsilon/2\} + P\{\max_{1\le\ell\le n} X_{\ell+[\sqrt{n}]} > \sqrt{n}\,\epsilon \mid \max_{1\le\ell\le\sqrt{n}} Y_\ell \le \sqrt{n}\,\epsilon/2\} \\ &\le P\{\max_{1\le\ell\le\sqrt{n}} Y_\ell > \sqrt{n}\,\epsilon/2\} + P\{\max_{1\le\ell\le n} Y'_\ell > \sqrt{n}\,\epsilon/2 + nq\} , \end{aligned}$$

where $Y'_\ell = Y_{\ell+[\sqrt{n}]} - Y_{[\sqrt{n}]}$. The last inequality results from

$$X_{\ell+\sqrt{n}} = \sum_{j=1}^{\ell} X_{j+[\sqrt{n}]} + X_{[\sqrt{n}]}$$

$$\leq Y'_\ell + Y_{[\sqrt{n}]} - \sqrt{n}\, q$$

$$\leq Y'_\ell + \sqrt{n}\,(\epsilon/2 - q) .$$

Using the maximal inequality as in Proposition 5,

$$P\{\max_{1 \leq \ell \leq \sqrt{n}} Y_\ell > \sqrt{n}\,\epsilon/2\} \leq 4c/\sqrt{n}^{\,2}$$

and

$$P\{\max_{1 \leq \ell \leq n} Y_\ell > nq\} \leq c/nq^2$$

We can simultaneously make these as small as we like, and the proposition follows.

CHAPTER 3

THE RANDOM WALK

In this chapter we define a type of random walk on the integer lattice of $\mathbb{R}^K$. The random walk is defined in Section 1. In Section 2 we introduce a mapping which transforms the random walk in $\mathbb{R}^K$ into a 'reflected random walk' (RRW) in $\mathbb{R}^K_+$. We then consider a sequence of random walks and RRW's in Section 3, proving limit theorems for both sequences. These results will be related to queueing networks in the next chapter.

3.1. The Random Walk

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which are defined mutually independent sequences of i.i.d. random variables $\{u_k^i, i \geq 1\}$, $\{v_k^i, i \geq 1\}$, and $\{\varphi_k^i, i \geq 1\}$ for $k = 1, \ldots, K$. The u_k^i's and v_k^i's are nonnegative and the φ_k^i's have support on $\{0, 1, \ldots, K\}$. We define (for $1 \leq k \leq K$),

$$\mu_k = 1/E[v_k^i] > 0 , \qquad s_k = \mathrm{var}(v_k^i) > 0 ,$$

$$\lambda_k = 1/E[u_k^i] \geq 0, \qquad a_k = \mathrm{var}(u_k^i) \geq 0 ,$$

with all of these terms assumed finite. In addition, we only allow $a_k = 0$ when $\lambda_k = 0$. We also have

$$p_{k\ell} = P\{\varphi_k^i = \ell\}, \quad 1 \leq \ell \leq K, \quad 1 \leq k \leq K .$$

The $K \times K$ matrix $P = \{p_{k\ell}, 1 \leq k \leq K, 1 \leq \ell \leq K\}$ is assumed to be substochastic in the sense that no square submatrix is stochastic.

When we define the queueing network, the u_k^i's will function as interarrival times to station k, the v_k^i's will function as service times for station k, and the φ_k^i's will function as routing indicators for customers served at station k. The matrix P is therefore called the routing matrix. The situation where $\lambda_k = 0$ corresponds to station k having no arrival stream from outside the network.

We now construct renewal processes in terms of the i.i.d. sequences defined above. Let

$$\alpha_k(0) = 0 , \qquad \alpha_k(\ell) = \sum_{i=1}^{\ell} u_k^i , \qquad \ell \geq 1,$$

$$\sigma_k(0) = 0 , \qquad \sigma_k(\ell) = \sum_{i=1}^{\ell} v_k^i , \qquad \ell \geq 1 .$$

We define

$$A_k(t) = \sup\{\ell \geq 0 : \alpha_k(\ell) \leq t\} , \qquad 1 \leq k \leq K, \ t \geq 0 ,$$

$$S_k(t) = \sup\{\ell \geq 0 : \sigma_k(\ell) \leq t\} , \qquad 1 \leq k \leq K, \ t \geq 0 .$$

Routing vectors are defined by

$$\underset{\sim}{\Phi}_k^i = e_{\varphi_k^i} ,$$

where e_j is a vector with a one in the jth position.

Let

$$\mathcal{A}_k = \{\alpha_k(\ell),\ \ell \geq 1\} \quad \text{for} \quad 1 \leq k \leq K ,$$

$$\mathcal{S}_k = \{\sigma_k(\ell),\ \ell \geq 1\} \quad \text{for} \quad 1 \leq k \leq K ,$$

$$\mathcal{A} = \bigcup_{k=1}^{K} \mathcal{A}_k , \qquad \mathcal{S} = \bigcup_{k=1}^{K} \mathcal{S}_k , \qquad \text{and} \qquad \mathcal{T} = \mathcal{A} \cup \mathcal{S}.$$

Let the following three sequences be enumerations of $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{T}$ respectively:

$$\alpha(1) \leq \alpha(2) \leq \cdots ,$$

$$\sigma(1) \leq \sigma(2) \leq \cdots .$$

and

$$t(1) \leq t(2) \leq \cdots .$$

We define the processes $S_{kj} = \{S_{kj}(t),\ t \geq 0\}$ for $1 \leq j \leq K$ as the components of the vector process

$$\underset{\sim}{S}_k(t) = \sum_{i=1}^{S_k(t)} \underset{\sim}{\Phi}_k^i .$$

Let $\underset{\rightarrow}{S}_k = (S_k, \underset{\sim}{S}_k)$ and $\underset{\rightarrow}{S} = (\underset{\rightarrow}{S}_1, \ldots , \underset{\rightarrow}{S}_K)$. We can now define our random walk. Let

$$(1) \qquad X_k(t) = A_k(t) + \sum_{\ell=1}^{K} S_{\ell k}(t) - S_k(t) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad 1 \leq k \leq K.$$

Let $\underset{\sim}{X} = \{\underset{\sim}{X}(t),\ t \geq 0\}$ be the vector process whose component processes are the $X_k = \{X_k(t),\ t \geq 0\}$. We call this a random walk, although this usage stretches standard terminology a bit. Let

(2) $$\nu_k = \lambda_k + \sum_{\ell=1}^{K} p_{\ell k}\mu_\ell .$$

The vector $\underset{\sim}{\nu}-\underset{\sim}{\mu}$ is, loosely speaking, the drift of $\underset{\sim}{X}$.

We construct an increasing family of sub σ-fields $\{\mathcal{F}_t, t \geq 0\}$ by setting

$$\mathcal{F}_t = \sigma\{A_k(u), S_k(u), \varphi^k_{i_k}; 1 \leq k \leq K, 0 \leq u \leq t, 1 \leq i_k \leq S_k(t)\}$$

for all $t \geq 0$. We can define a related increasing sequence of sub σ-fields $\{\bar{\mathcal{F}}_i, i \geq 0\}$ by

$$\bar{\mathcal{F}}_i = \mathcal{F}_{t(i)} ,$$

The sample paths of $\underset{\sim}{X}$ are elements of D^K. At certain points in the development it is easier to deal with the related process in C^K obtained from $\underset{\sim}{X}$ by linearly interpolating between the jumps. Toward this end, let

(3) $$\underset{\sim}{W}(t) = \frac{t(\ell+1)-t}{t(\ell+1)-t(\ell)}\underset{\sim}{X}(t(\ell)) + \frac{t - t(\ell)}{t(\ell+1)-t(\ell)}\underset{\sim}{X}(t(\ell+1))$$

for $t(\ell) \leq t < t(\ell+1)$.

The paths of $\underset{\sim}{W}$ are piecewise linear.

3.2. The Reflection Mapping

We now present a mapping, which we call f, that transforms a random walk of the type described in the last section to a reflected random walk (RRW) which stays inside of $\mathbb{R}^K_+$. It is no longer restricted to the integer lattice, however. The mapping f was described informally in Section 1.4.

Let D^K_+ be the subset of D^K consisting of functions which are nonnegative and nondecreasing. For $x \in D^K$, let

$$(4) \qquad U(x) = \{y \in D^K_+ | x(t) + y(t)[I-P] \geq 0 \quad \text{for all} \quad t \geq 0\}$$

Proposition 1. For each $x \in D^K$, $U(x)$ is nonempty.

Proof. We explicitly display an element of U. Let

$$\underset{\sim}{y}^{o}(t) = \underset{\sim}{m}(t)R \qquad \text{for} \quad t \geq 0 \; .$$

where

$$m_k(t) = - \inf_{0 \leq s \leq t} \{x_k(s)\} \wedge 0 \; , \qquad k = 1,\ldots,K \quad \text{and} \quad t \geq 0,$$

and $R = (I-P)^{-1}$, as described in Section 1.2. We have

$$\underset{\sim}{y}^{o}(t)[I-P] = \underset{\sim}{m}(t) \; ,$$

or

$$y^o_k(t) - \sum_{j=1}^{k} p_{jk} y^o_j(t) = m_k(t) \; , \qquad k = 1,\ldots,K \; .$$

Since $m_k(t) \geq -x_k(t)$ we have

$$x_k(t) + y_k^o(t) - \sum_{j=1}^{k} p_{jk} y_j^o(t) \geq 0 \ , \qquad k = 1,\ldots,K \ ,$$

for all $t \geq 0$. So $\underset{\sim}{\boldsymbol{y}}^o \in U(x)$.

We say that $\underset{\sim}{y} \in U(\underset{\sim}{x})$ is a minimal element of $U(\underset{\sim}{x})$ if $\boldsymbol{y}_k(t) \leq y_k(t)$ for $1 \leq k \leq K$, $t \geq 0$, for all $\underset{\sim}{y} \in U(\underset{\sim}{x})$. It is clear that if such a minimal element exists then it is unique.

<u>Proposition 2</u>. For each $\underset{\sim}{x} \in D^K$, $U(x)$ has a minimal element.

<u>Proof</u>. Since U is defined in terms of weak inequalities it is clearly closed. Suppose $\underset{\sim}{y}' = \{\underset{\sim}{y}(t),\ t \geq 0\} \in U$ and $\underset{\sim}{y}'' = \{\underset{\sim}{y}''(t),\ t \geq 0\} \in U$. We need to show that

$$\underset{\sim}{y}^* = \{(y_1'(t) \wedge y_1''(t),\ \ldots\ ,\ y_K'(t) \wedge y_K''(t)),\ t \geq 0\} \in U \ .$$

There are 2^K possible cases, depending on whether $y_k'(t) \geq y_k''(t)$ or $y_k''(t) > y_k'(t)$. It suffices to show the result when $\underset{\sim}{y}'(t)$ and $\underset{\sim}{y}''(t)$ differ in only two components, since we can get the general case with several applications of this special case. Without loss of generality we can assume that the first two components of $\underset{\sim}{y}'(t)$ and $\underset{\sim}{y}''(t)$ are different and the rest are the same. There are four cases to consider:

i)	$y_1'(t) \leq y_1''(t)$	and	$y_2'(t) \leq y_2''(t)$,
ii)	$y_2'(t) > y_1''(t)$	and	$y_2'(t) > y_2''(t)$,
iii)	$y_1'(t) \leq y_1''(t)$	and	$y_2'(t) > y_2''(t)$,
iv)	$y_1'(t) > y_1''(t)$	and	$y_2'(t) \leq y_2''(t)$.

For case i) $\underset{\sim}{y}^*(t) = \underset{\sim}{y}'(t)$, and for case ii) $\underset{\sim}{y}^*(t) = \underset{\sim}{y}''(t)$. Cases iii) and iv) are symmetric so we deal only with iii).

We need to show

$$(5) \qquad x_1(t) + y_1'(t) - p_{21}y_2''(t) - \sum_{j=3}^{K} p_{j1}y_j'(t) \geq 0$$

$$(6) \qquad x_2(t) + y_2''(t) - p_{12}y_1'(t) - \sum_{j=3}^{K} p_{j2}y_j''(t) \geq 0$$

$$(7) \qquad x_k(t) + y_k''(t) - p_{1k}y_1'(t) - p_{2k}y_2''(t) - \sum_{j=3}^{K} p_{jk}y_j''(t) \geq 0,$$

for $k = 3,\ldots,K$.

From (iii) we have

$$(8) \qquad x_1(t) + y_1'(t) - p_{21}y_2''(t) - \sum_{j=3}^{K} p_{j1}y_j'(t)$$

$$\geq x_1(t) + y_1'(t) - p_{21}y_2'(t) - \sum_{j=3}^{K} p_{j1}y_j'(t)$$

$$(9) \qquad x_2(t) + y_2''(t) - p_{12}y_1'(t) - \sum_{j=3}^{K} p_{j2}y_j''(t)$$

$$\geq x_2(t) + y_2''(t) - p_{12}y_1''(t) - \sum_{j=3}^{K} p_{j2}y_j''(t)$$

$$(10) \qquad x_k(t) + y_k''(t) - p_{1k}y_1'(t) - p_{2k}y_2''(t) - \sum_{j=3}^{K} y_j''(t)$$

$$\geq x_k(t) + y_k''(t) - \sum_{j=1}^{K} p_{jk}y_j''(t), \qquad k = 1,\ldots,K.$$

The expressions on the right-hand side in (8)-(10) are nonnegative, since $\underset{\sim}{y}', \underset{\sim}{y}'' \in U(\underset{\sim}{x})$. Thus we have (5)-(7), completing the proof.

In the next section we will apply the continuous mapping theorem to obtain weak convergence for a sequence of RRW's. To do this we need the following result.

Proposition 3. $f: C^K \to C^K$ is continuous.

Proof. By the definition of continuity, we must show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, x' \in C^K$, if $\rho(\underset{\sim}{x},\underset{\sim}{x}') < \delta$ then $\rho(f(\underset{\sim}{x}),f(\underset{\sim}{x}')) < \epsilon$.

Let $\underset{\sim}{y}$ be the minimal element of $U(\underset{\sim}{x})$ and let $\underset{\sim}{y}'$ be the minimal element of $U(\underset{\sim}{x}')$. Take $\epsilon > 0$ to be given. Showing that $\rho(\underset{\sim}{y},\underset{\sim}{y}') < \epsilon$ is enough. Choose δ so that $|\delta R\underset{\sim}{1}| < \epsilon$ where $\underset{\sim}{1}$ is a vector with a one in every position and $|\cdot|$ here denotes the Euclidean norm. If $\rho(\underset{\sim}{x},\underset{\sim}{x}') < \delta$, then

$$\hat{\underset{\sim}{y}} = \underset{\sim}{y} + \delta R1 \in U(\underset{\sim}{x}')$$
$$\hat{\underset{\sim}{y}}' = \underset{\sim}{y}' + \delta R1 \in U(x) \; .$$

We show that $\hat{\underset{\sim}{y}}' \in U(\underset{\sim}{x})$, the other case is similar. We have

$$\hat{\underset{\sim}{z}}'(t) = \underset{\sim}{x}(t) + \hat{\underset{\sim}{y}}'(t)[I-P] \geq \underset{\sim}{x}'(t) - \delta 1 + \underset{\sim}{y}'(t)[I-P] + \delta 1 R[I-P]$$
$$= \underset{\sim}{x}'(t) + \underset{\sim}{y}'(t)[I-P] \geq 0 \; .$$

The first inequality follows from the definition of $\hat{\underset{\sim}{y}}'$ and the fact that $\rho(\underset{\sim}{x},\underset{\sim}{x}') < \delta$. The equality follows from the fact that $R = [I-P]^{-1}$, and the final inequality from the assumption that $\underset{\sim}{y}' \in U(x')$.

From the minimality definition of $\underset{\sim}{y}$ and $\underset{\sim}{y}'$ we have

$$\underset{\sim}{y} - \underset{\sim}{y}' \leq \hat{\underset{\sim}{y}}' - \underset{\sim}{y}' = \delta 1R$$

$$\underset{\sim}{y}' - \underset{\sim}{y} \leq \hat{\underset{\sim}{y}} - \underset{\sim}{y} = \delta 1R\ ,$$

so that

$$\rho(\underset{\sim}{y},\underset{\sim}{y}') \leq |\delta 1R| \leq \epsilon$$

3. Limit Theorems for Random Walk and RRW

We now turn our attention to a sequence of random walks of the type described in Section 1. We thus consider a sequence of probability spaces $\{(\Omega^n, \mathfrak{F}^n, P^n),\ n \geq 1\}$, each element of which is similar to $(\Omega, \mathfrak{F}, P)$ of Section 1 to the extent that it has sequences of random variables as described in Section 1 with which we can construct a random walk. We carry forward all of the notation established in Sections 1 and 2 except that all quantities which depend on n will have an n appended in a convenient place to signify this dependence. Both K and P remain fixed, so they are not subscripted. Let

$$\underset{\sim}{X}^{\#n}(t) = n^{-1/2}\, \underset{\sim}{X}^n(nt) \quad \text{for } 0 \leq t \leq 1 \text{ and } n \geq 1.$$

Theorem 1. Suppose

i) $\mu_k(n) \to \mu_k$ as $n \to \infty$ for $1 \leq k \leq K$,

ii) $\lambda_k(n) \to \lambda_k$ as $n \to \infty$ for $1 \leq k \leq K$,

iii) $a_k(n) \to a_k$ as $n \to \infty$ for $1 \leq k \leq K$,

iv) $s_k(n) \to s_k$ as $n \to \infty$ for $1 \leq k \leq K$,

v) $(\nu_k(n) - \mu_k(n))\sqrt{n} \equiv c_k^n \to c_k$ as $n \to \infty$, $-\infty < c_k < \infty$ for $1 \leq k \leq K$,

vi) $\max\limits_{1 \leq k \leq K} \sup\limits_{n \geq 1} E\{(v_k^1(n))^{2+\epsilon}\} < \infty$ for some $\epsilon > 0$,

and

vii) $\max\limits_{1 \leq k \leq K} \sup\limits_{n \geq 1} E\{(u_k^1(n))^{2+\epsilon}\} < \infty$ for some $\epsilon > 0$.

Then $\underset{\sim}{X}^{*n} \Rightarrow \underset{\sim}{X}^*$ in D^K as $n \to \infty$, where $\underset{\sim}{X}^*$ is K dimensional Brownian motion with drift vector $\underset{\sim}{c}$ and covariance matrix

$$\ddagger = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 & & \sigma_{1K}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 & & \\ \vdots & & \ddots & \vdots \\ \sigma_{1K}^2 & \cdots & \cdots & \sigma_{KK}^2 \end{pmatrix},$$

where

(11) $$\sigma_{ii}^2 = \lambda_i^3 a_i + \mu_i^3 s_i + \sum_{\ell=1}^{k} \mu_\ell p_{\ell k}(1 - p_{\ell k} + p_{\ell k}\mu_\ell^2 s_\ell) \quad \text{for } 1 \leq i \leq K$$

and

(12) $$\sigma_{ij}^2 = -[\mu_i^3 s_i p_{ij} + \mu_j^3 s_j p_{ji} - \sum_{\ell=1}^{k} \mu_\ell p_{\ell i} p_{\ell j}(1 - \mu_\ell^2 s_\ell) \quad \text{for } 1 \leq i < j \leq K.$$

Proof. Let $A_k^{\#n}(t) = n^{-1/2}[A_k^n(nt) - \lambda_k(n)nt]$. Using ii), iii) and vii), Proposition 2.7 says that $A_k^{\#n} \Rightarrow \sqrt{\lambda_k^3 u_k}\,\beta_k$ in D as $n \to \infty$, for $k = 1,\ldots,K$, where $\beta_1,\ldots,\beta_K$ are independent standard Brownian motions.

Let

$$S_k^{\#n}(t) = n^{-1/2}[S_k^n(nt) - \mu_k(n)nt],$$

$$\underset{\sim}{S}_k^{\#n}(t) = n^{-1/2} \sum_{\ell=1}^{S_k^n(nt)} (\underset{\sim}{\Phi}_k^\ell - p_{k.}) ,$$

and

$$\bar{\underset{\sim}{S}}_k^{\#n}(t) = n^{-1/2} \sum_{\ell=1}^{[\mu_k nt]} (\underset{\sim}{\Phi}_k^\ell - p_{k.}) \quad \text{for } 1 \le k \le K,\ 0 \le t \le 1 \text{ and } n \ge 1.$$

Using i), iv), vi) and Proposition 2.7 once again, we have

$S_k^{\#n} \Rightarrow \sqrt{\mu_k^3 s_k}\,\xi_k$ in D as $n \to \infty$. We claim that $\rho(\underset{\sim}{S}_k^{\#n}, \bar{\underset{\sim}{S}}_k^{\#n}) \Rightarrow 0$ as $n \to \infty$. We have .

$$|\underset{\sim}{S}_k^{\#n}(t) - \bar{\underset{\sim}{S}}_k^{\#n}(t)| = n^{-1/2} \sum_{\ell=S_k^n(nt)\wedge[\mu_k nt]}^{S_k^n(nt)\vee[\mu_k nt]} (\underset{\sim}{\Phi}_k^\ell - p_{k.})$$

$$\overset{d}{=} n^{-1/2} \sum_{\ell=1}^{|S_k^n(nt)-[\mu_k nt]|} (\underset{\sim}{\Phi}_k^\ell - p_{k.})$$

$$\le n^{-1/2} \sum_{\ell=1}^{[|\mu_k(n)nt-\mu_k nt|]+|S_k^n(nt)-[\mu_k(n)nt]|} (\Phi_k^\ell - p_{k.}) .$$

First look at

$$n^{-1/2} \sum_{\ell=1}^{[|\mu_n(n)nt-\mu_k nt|]} (\Phi_k^\ell - p_{k.}) .$$

By hypothesis, this is close to

$$n^{-1/2} \sum_{\ell=1}^{[|c_k|\sqrt{n}\, t]} (\Phi_k^{\ell} - p_{k.}) ,$$

which, by Kolmogorov's inequality converges to zero in probability uniformly on $0 \leq t \leq 1$. Now renumber the sequence and consider

$$n^{-1/2} \sum_{\ell=1}^{|S_k^n(nt) - \mu_k(n)nt|} (\underset{\sim}{\Phi}_k^{\ell} - p_{k.}) = n^{-1/2} \sum_{\ell=1}^{[\sqrt{n}|S_k^{\#n}(t)|]} (\underset{\sim}{\Phi}_k^{\ell} - p_{k.}) .$$

Since $S_k^{\#n} \Rightarrow \sqrt{\mu_k^3 s_k}\, \xi_k$, we get this term to converge weakly to zero using Proposition 2.5.

We can now deal with $\bar{\underset{\sim}{S}}_k^{\#n}$ instead of $\underset{\sim}{S}_k^{\#n}$. This is helpful because $\bar{\underset{\sim}{S}}_k^{\#n}$ is independent of $S_k^{\#n}$. Let

$$I_k = \{1 \leq i \leq K | p_{ki} > 0\}$$

and

$$\bar{I}_k = \{1,\ldots,L\} \setminus I_k .$$

We use $[\bar{\underset{\sim}{S}}_k^{\#n}(t)]_{I_k}$ to denote the subvector of $\bar{\underset{\sim}{S}}_k^{\#n}(t)$ corresponding to indices in I_k. By Proposition 2.2, for $k = 1,\ldots,K$, we have

$$[\bar{\underset{\sim}{S}}_k^{\#}]_{I_k} \Rightarrow [\mu_k \underset{\sim}{\zeta}_k]_{I_k}, \quad \text{with} \quad [\underset{\sim}{\zeta}_k]_{\bar{I}_k} = 0 ,$$

where $\underset{\sim}{\zeta}_k$ are independent K dimensional Brownian motions with zero drift vectors and covariance matrices $\ddagger_k$.

$$(\Sigma_k)_{ij} = \begin{cases} -p_{ki}p_{kj} & i \neq j. \\ p_{ki}(1-p_{ki}) & i = j . \end{cases}$$

It should be clear that $(\Sigma_k)_{ij} = 0$ if either $i \in \bar{I}_k$ or $j \in \bar{I}_k$.

By the independence of $S_k^{\#n}$ and $\bar{\underset{\sim}{S}}_k^{\#n}$ we have

$$(S_k^{\#n}, \bar{\underset{\sim}{S}}_k^{\#n}) \Rightarrow (\sqrt{\mu_k^3 s_k}\ \xi_k, \ \mu_k \underset{\sim}{\zeta}_k) \quad \text{in} \quad D^{K+1} \quad \text{as} \quad n \to \infty ,$$

for $1 \leq k \leq K$. The converging together theorem gives us

$$(S_k^{\#n}, \underset{\sim}{S}_k^{\#n}) \Rightarrow (\sqrt{\mu_k^3 s_k}\ \xi_k, \ \mu_k \underset{\sim}{\zeta}_k) \quad \text{in} \quad D^{K+1} \quad \text{as} \quad n \to \infty ,$$

for $1 \leq k \leq K$.

Since

$$X_k^n(nt) = A_k^n(nt) + \sum_{\ell=1}^{K} S_{\ell k}^n(nt) - S_k^n(nt) ,$$

we have

$$X_k^{\#n}(t) = A_k^{\#n}(t) + \lambda_k(n) \sqrt{n}\ t + \sum_{\ell=1}^{K} (S_{\ell k}^{\#n}(t) + p_{\ell k} S_\ell^{\#n}(t) + p_{\ell k}\mu_\ell(n) \sqrt{n}\ t)$$

$$- S_k^{\#n}(t) - \mu_k(n) \sqrt{n}\ t .$$

Using the independence of $A_k^{\#n}$ and $(S_k^{\#n}, \underset{\sim}{S}_k^{\#n})$ for $k = 1,\ldots,K$, as well as the independence of all processes with different subscripts, we obtain

(13) $$\underset{\sim}{X}^{*n}(\cdot) \Rightarrow (\sqrt{\lambda_1^3 a_1}\, \beta_1(\cdot), \ldots, \sqrt{\lambda_K^3 a_K}\, \beta_K(\cdot)$$

$$+ \sum_{\ell=1}^{K} (\sqrt{\mu_\ell}\, \underset{\sim}{\zeta}_\ell(\cdot) + \sqrt{\mu_\ell^3 s_\ell}\, \xi_\ell(\cdot) \underset{\sim}{p}_{\ell\cdot})$$

$$- (\sqrt{\mu_1^3 s_1}\, \xi_1(\cdot), \ldots, \sqrt{\mu_K^3 s_K}\, \xi_K(\cdot)) + \underset{\sim}{c} \quad ,$$

from the continuous mapping theorem. The process on the right in (13) is equal in distribution to a K dimensional Brownian motion $X^* = \{\underset{\sim}{X}^*(t),\ t \geq 0\}$ with drift vector $\underset{\sim}{c}$ and covariance $\ddagger$,

$$\ddagger = \begin{pmatrix} \sigma_{11}^2 & -\sigma_{12}^2 & & & -\sigma_{1K}^2 \\ -\sigma_{12}^2 & \sigma_{22}^2 & -\sigma_{23}^2 & & \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \sigma_{1K}^2 & \cdots & \cdots & \cdots & \sigma_{KK}^2 \end{pmatrix}$$

with

$$\sigma_i^2 = \lambda_i^3 a_i + \mu_i^3 s_i + \sum_{\ell=1}^{K} \mu_\ell p_{\ell k}(1 - p_{\ell k} + p_{\ell k}\mu_\ell^2 s_\ell)$$

and
$$\sigma_{ij}^2 = \mu_i^3 s_i p_{ij} + \mu_j^3 s_j p_{ji} - \sum_{\ell=1}^{K} \mu_\ell p_{\ell i} p_{\ell j}(1 - \mu_\ell^2 s_\ell) \ .$$

This completes the proof.

Let $\underset{\sim}{W}^{*n}(t) = n^{-1/2} \underset{\sim}{W}^n(nt)$ for $n \geq 1,\ 0 \leq t \leq 1$.

Corollary 1. Under the hypotheses of Theorem 1, $\underset{\sim}{W}^{*n} \Rightarrow \underset{\sim}{X}^*$ in C^K as $n \to \infty$.

Proof. By the definition of $\underset{\sim}{W}$, $\rho(\underset{\sim}{W}^{*n}, \underset{\sim}{X}^{*n}) \leq 2/\sqrt{n}$ so $\rho(\underset{\sim}{W}^{*n}, \underset{\sim}{X}^{*n}) \Rightarrow 0$ as $n \to \infty$. Using Proposition 2.1 we have the result.

Let $\underset{\sim}{Z}^{*n}(t) = n^{-1/2}\underset{\sim}{Z}^n(nt)$ for $n \geq 1$ and $0 \leq t \leq 1$, and let $\underset{\sim}{Z}^* = f(\underset{\sim}{X}^*)$.

Theorem 2. Under the hypotheses of Theorem 1, $\underset{\sim}{Z}^{*n} \Rightarrow \underset{\sim}{Z}^*$ in D^K as $n \to \infty$.

Proof. Using the continuous mapping theorem we get $f(\underset{\sim}{W}^{*n}) \Rightarrow f(\underset{\sim}{X}^*)$ as $n \to \infty$. The continuity of f gives $\rho(f(\underset{\sim}{W}^{*n}), \underset{\sim}{Z}^{*n}) \Rightarrow 0$ as $n \to \infty$ from $\rho(\underset{\sim}{W}^{*n}, \underset{\sim}{X}^{*n}) \Rightarrow 0$ as $n \to \infty$. Applying Proposition 2.1 yields the result.

CHAPTER 4

QUEUEING NETWORKS

To prove the limit theorem for queueing networks, we need to construct the vector queue length process on $(\Omega, \mathcal{F}, P)$, in terms of the sequence of random variables defined in Section 3.1. In addition, we will require several modified queueing networks to aid in the proof. The vector queue length processes of the modified networks are also constructed on $(\Omega, \mathcal{F}, P)$. The standard network and all of is modifications are defined in this chapter.

4.1. The Bororkov Network

The queueing network we introduce in this section is closely related to the modification for GI/G/S queues first introduced by Borovkov (1965), and is called the Borovkov network. We call the vector queue length process for the Borovkov network the Borovkov process.

The Borovkov network has K stations, each of which functions as a Bororkov modified single server queue. The process A_k is the input process to station k $(k = 1,\ldots,K)$ from the outside of the network, so $A_k(t)$ denotes the number of customers who enter station k from outside of the network during $[0,t]$. The process S_k is the potential service process for station k $(k = 1,\ldots,K)$. As long as all stations are busy (nonempty), the successive potential service times act as real service times. When a customer arrives to an empty station he departs from that station at the completion of the potential service time in progress at the moment of his arrival. Potential service

completions at an empty station have no effect on the Borovkov process. A customer who leaves station k at the completion of the potential service time v_k^i is routed to the station indicated by φ_k^i. If $\varphi_k^i = 0$ then the customer leaves the network.

We denote the subsequence of potential service times at station k which result in customer departures by $v_k^{\ell_i}$ and the associated routing indicators by $\varphi_k^{\ell_i}$. Let $\bar{S}_k(t)$ count the number of customer departures from station k $(1 \leq k \leq K)$ during $[0,t]$. In addition, let $\bar{S}_{jk}(t)$ denote the number of customers routed directly from station j to station k $(1 \leq j,\ k \leq K)$ during $[0,t]$. We then have

$$\bar{S}_{jk}(t) = \begin{cases} \sum_{i=1}^{\bar{S}_j(t)} \Phi_{jk}^{\ell_i} & \text{if } \bar{S}_j(t) \geq 1 \\ 0 & \text{if } \bar{S}_j(t) = 0\ , \end{cases}$$

for $1 \leq j,\ k \leq K$ and $t \geq 0$. Recall that $\underset{\sim}{\Phi}_j^{\ell_i} = \underset{\sim}{e}_{\varphi_j^{\ell_i}}$, where $\underset{\sim}{e}_i$ is the K vector with one in the ith position and zeroes everywhere else. We let $\bar{\underset{\sim}{Q}} = \{\bar{\underset{\sim}{Q}}(t),\ t \geq 0\}$ be the Borovkov process, so we have

$$(1) \qquad \bar{Q}_k(t) = A_k(t) + \sum_{j=1}^{K} \bar{S}_{jk}(t) - \bar{S}_k(t) \quad \text{for } 1 \leq k \leq K \text{ and } t \geq 0.$$

Let $\bar{A}_k(t) = A_k(t) + \sum_{j=1}^{K} \bar{S}_{jk}(t)$. Because of the way the Borovkov network operates (Iglehart and Whitt (1970a)) we have

(2) $\bar{Q}_k(t) = [\bar{A}_k(t) - S_k(t)] - \inf_{0 \le s \le t} \{\bar{A}_k(s) - S_k(s)\}$

for $i \le k \le K$ and $t \ge 0$. Thus

(3) $\bar{S}_k(t) = A_k(t) - \bar{Q}_k(t) = S_k(t) + \inf_{0 \le s \le t} \{\bar{A}_k(s) - S_k(s)\}$.

4.2. The Standard Network

The standard queueing network can be defined on $(\Omega, \mathcal{F}, P)$ and related to the Borovkov network. The vector queue length process for the network defined here is distributionally equivalent to that of the network originally defined in Section 1.2.

Each of the K stations of the standard network operates as a standard single server queue in conjunction with the identically indexed station in the Borovkov network. Thus if station k $(1 \le k \le K)$ in the standard network is nonempty at the completion of service, the next service time is the next element in the potential service sequence for station k. When a customer arrives to station k at time $t \ge 0$ and finds it empty, his service time is the next unused (by the standard network) potential service time occurring after t in S_k. To make this verbal description precise, let $N_k(t)$ be the index of the last potential service time used by station k in the standard network by time $t \ge 0$. The index of the potential service time used to serve the next customer at station k (who begins service at time t) is then $\max(N_k(t), S_k(t) + 1) + 1$. This procedure clearly yields an

i.i.d. sequence. The routing indicators used in the standard network are the same as those used in the Borovkov network. The ith customer served at station k in the standard network is routed by $\varphi_k^{\ell_i}$. In particular, it is not necessarily true that the index of the routing indicator used for a customer in the standard network matches the index of his potential service time. The sequence of routing indicators is i.i.d. and independent of the service times. The arrival processes from outside the standard network are the same as those for the Borovkov network.

We let $\hat{S}_k(t)$ denote the number of customers served at station k during $[0,t]$, and let $\hat{S}_{kj}(t)$ denote the number of customers routed from station k to station j during $[0,t]$. Defining $\underset{\sim}{Q} = \{Q(t),\ t \geq 0\}$ as the vector queue length process for the standard network, we have

$$(4) \quad Q_k(t) = A_k(t) + \sum_{j=1}^{K} \hat{S}_{jk}(t) - \hat{S}_k(t) \quad \text{for} \quad 1 \leq k \leq K \quad \text{and} \quad t \geq 0.$$

We now define another vector process associated with the standard network which will be useful in the proof of the limit theorem. Let $m(j,k,\ell)$ be the number of visits to station k made by the jth customer arriving to station ℓ from outside the network for $1 \leq k,\ \ell \leq K$ and $j \geq 1$. Then

$$\{m(j,k,\ell),\ j \geq 1\}, \quad 1 \leq k,\ \ell \leq K,$$

are mutually independent sequences of i.i.d. random variables. Since the sequence of stations visited by a customer forms a Markov chain (the customer is absorbed in state 0 when he exits the network), $m(j,k,\ell)$ has a geometric distribution with $E[m(j,k,\ell)] = R_{k\ell}$. We define the process $\underset{\sim}{N} = \{\underset{\sim}{N}(t),\ t \geq 0\}$ by

$$(5) \qquad N_\ell(t) = \sum_{k=1}^{K} \sum_{j=1}^{A_k(t)} m(j,k,\ell) \qquad \text{for } 1 \leq \ell \leq K \text{ and } t \geq 0 ,$$

and define $\underset{\sim}{L} = \{\underset{\sim}{L}(t),\ t \geq 0\}$ by

$$(6) \qquad L_\ell(t) = N_\ell(t) - \hat{S}_\ell(t) \qquad \text{for } 1 \leq \ell \leq K \text{ and } t \geq 0.$$

We call N_ℓ the total load process for station ℓ and L_ℓ the load process for station ℓ.

There is a relation between the load process and queue length process. We have

$$(7) \qquad E[L_\ell(t) \mid \mathcal{F}_t] = \sum_{j=1}^{K} Q_j(t)\, R_{j\ell} , \qquad \text{for } i \leq \ell \leq K \text{ and } t \geq 0 ,$$

from which we get

$$(8) \qquad \max_{1 \leq j \leq K} Q_j(t) \geq \max_{1 \leq \ell \leq K} [K \max_{1 \leq j \leq K} R_{j\ell}]^{-1}\, E[L_\ell(t) \mid \mathcal{F}_t] .$$

4.3. The Input-Output Network

We now turn to the Input-Output (I/O) network, which is related to both the Borovkov and standard networks. To understand how this modification works, refer to Figure 3, which depicts the I/O network for the two station case.

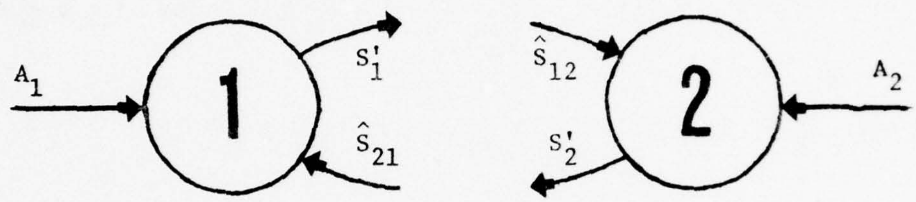

FIGURE 3: The Input-Output Network

The network is constructed in such a manner that it can be viewed as K separate Borovkov modified single server queues. The input processes from outside the network are again A_k $(1 \leq k \leq K)$. In addition, station k $(1 \leq k \leq K)$ receives inputs $\hat{S}_{jk}$ for $1 \leq j \leq K$. Thus the net input to a station in the I/O network is equal path by path to the net input for the identically indexed station in the standard network. Station k $(1 \leq k \leq K)$ uses S_k as its potential service process. All customers completing service immediately leave the network. Let $S_k'(t)$ denote the number of customers served at station k $(1 \leq k \leq K)$ during $[0,t]$ in the I/O network, and let $Q_k'(t)$ denote the number of customers in station k at time t for the I/O network. Then

$$Q'_k(t) = A_k(t) + \sum_{j=1}^{K} \hat{S}_{jk}(t) - S'_k(t) \quad \text{for} \quad 1 \leq k \leq K \quad \text{and} \quad t \geq 0 .$$

Since each station operates as a Borovkov modified queue, we have

$$(10) \qquad S'_k(t) = S_k(t) + \inf_{0 \leq S \leq t} \{A'_k(t) - S_k(t)\} \quad \text{for} \quad 1 \leq k \leq K \quad \text{and} \quad t \geq 0,$$

where $A'_k(t) = A_k(t) + \sum_{j=1}^{K} \hat{S}_{jk}(t)$ for $1 \leq k \leq K$ and $t \geq 0$. Let

$$\underset{\sim}{S}'_k = (S'_{k1}, S'_{k2}, \ldots , S'_{kK}) \quad \text{for} \quad 1 \leq k \leq K ,$$

$$\underset{\sim}{\bar{S}}_k = (\bar{S}_{k1}, \ldots , \bar{S}_{kK}) \quad \text{for} \quad 1 \leq k \leq K ,$$

$$\underset{\rightarrow}{S}'_k = (S'_k, \underset{\sim}{S}'_k) \quad \text{for} \quad 1 \leq k \leq K ,$$

$$\underset{\rightarrow}{\bar{S}}_k = (\bar{S}_k, \underset{\sim}{\bar{S}}_k) \quad \text{for} \quad 1 \leq k \leq K ,$$

$$\underset{\rightarrow}{S}' = (\underset{\rightarrow}{S}'_1, \ldots , \underset{\rightarrow}{S}'_K) ,$$

$$\underset{\rightarrow}{\bar{S}} = (\underset{\rightarrow}{\bar{S}}_1, \ldots , \underset{\rightarrow}{\bar{S}}_K) ,$$

$$\underset{\sim}{\varphi}_k = (\varphi^1_k, \varphi^2_k, \ldots) \quad \text{for} \quad 1 \leq k \leq K,$$

and

$$\underset{\rightarrow}{\varphi} = (\underset{\sim}{\varphi}_1, \ldots , \varphi_K) .$$

We define $h(\underset{\sim}{A}, \underset{\rightarrow}{S}, \underset{\rightarrow}{\hat{S}}, \underset{\rightarrow}{\varphi}) = \underset{\rightarrow}{S'}$ by (10) and

$$S'_{kj}(t) = \sum_{i=1}^{S'_k(t)} \phi_{kj}^{\ell_i} \qquad \text{for } 1 \leq j, k \leq K .$$

Let $\{0,1,\ldots,K\}^{K\times\infty}$ be the space which contains the entire routing sequence for paths of the random walk, endowed with the discrete topology. Then $h: D^{2K^2+K} \times \{0,1,\ldots,K\}^{K\times\infty} \to D^{K^2}$. When viewed as a map restricted to $C^{2K^2+K} \times \{0,1,\ldots,K\}^{K\times\infty}$, h maps into C^{K^2} and is continuous. This is because h is constructed with composition of continuous functions, which are again continuous.

It is helpful at this juncture to point out the relation between the Borovkov network and I.O. network using the function h. From (3) in Section 4.1 we have

$$\underset{\rightarrow}{\bar{S}} = h(\underset{\sim}{A}, \underset{\rightarrow}{S}, \underset{\rightarrow}{\bar{S}}, \underset{\rightarrow}{\varphi}) . \tag{11}$$

An important point to note is that, subject to $\underset{\rightarrow}{\bar{S}}(0) = 0$, the equation (11) has a unique solution, which is the vector process for the Borovkov network.

4.4. The Reflected Random Walk and the Borovkov Process

We again consider the RRW of Section 3.2 and show that, viewed on a 'local' level, it behaves very much like the Borovkov process. The jump times of both processes are elements of $\mathcal{J}$. The only difference

in behavior of the processes occurs when one or both encounter a boundary. At jumps associated with arrivals the processes move away from the boundaries so that both processes always change by the same amount for arrivals. Let

$$\Delta g(t) \equiv g(t) - g(t-) , \qquad \text{for } t \geq 0 ,$$

for $g \in D^K$. Then if $t \in \mathcal{A}\backslash\mathcal{S}$ we have

$$\Delta(Z(t) - \bar{Q}(t)) = 0 .$$

Thus we only need to deal with $t \in \mathcal{S}$. We begin by splitting $\mathcal{S}$ into several smaller sets according to the relative positions of the two processes. For $1 \leq k \leq K$, let

$$T_k = \{\sigma_k(\ell):Z_k(\sigma_k(\ell)-) = 0\} ,$$

$$T'_k = \{\sigma_k(\ell):0 < Z_k(\sigma_k(\ell)-) < 1\} ,$$

$$T''_k = \{\sigma_k(\ell):Z_k(\sigma_k(\ell)-) \geq 1\} ,$$

$$\bar{T}_k = \{\sigma_k(\ell):\bar{Q}_k(\sigma_k(\ell)-) = 0\} ,$$

and

$$\bar{T}'_k = \{\sigma_k(\ell):\bar{Q}_k(\sigma_k(\ell)-) > 0\} .$$

We then construct the following subsets of $\mathcal{S}$:

(for $1 \leq k \leq K$)

(i) $\mathcal{S}_k^1 = T_k \cap \bar{T}_k$

(ii) $\mathcal{S}_k^2 = T_k \cap \bar{T}'_k$,

(iii) $\mathcal{S}_k^3 \quad T'_k \cap \bar{T}_k$,

(iv) $\mathcal{S}_k^4 \quad T'_k \cap \bar{T}'_k$,

(v) $\mathcal{S}_k^5 \quad T''_k \cap \bar{T}_k$,

and

(vi) $\mathcal{S}_k^6 \quad T''_k \cap \bar{T}'_k$.

Let $\mathcal{S}^i = \bigcup_{k=1}^{K} \mathcal{S}_k^i$ for $1 \leq i \leq 6$. Then $\mathcal{S} = \bigcup_{i=1}^{6} \mathcal{S}^i$. Let

$$H(t) = \Delta\Big(\sum_{j=1}^{K} |\bar{Q}_j(t) - Z_j(t)| \Big) .$$

We now investigate how $H(t)$ behaves on each of the six sets above. Assume $t = \sigma_n(i) = \sigma(i') = t(i'')$ and $\varphi_k^i = j$. Let

$$b_\ell^{i'} = |\bar{Q}_\ell(t) - Z_\ell(t)|$$

and define

$$s_\ell^{i'}(x) = \begin{cases} x & \text{if } Z_\ell(t-) \geq \bar{Q}_\ell(t-) \text{ and } x \geq 0 \\ & \text{or } Z_\ell(t-) \geq \bar{Q}_\ell(t-)-x \text{ and } x \geq 0 \\ -x & \text{if } Z_\ell(t-) \leq \bar{Q}_\ell(t) \text{ and } x \leq 0 \\ & \text{or } Z_\ell(t-) \leq \bar{Q}_\ell(t-)-x \text{ and } x \geq 0 \\ |x|-2b_\ell^{i'} & \text{if } \bar{Q}_\ell(t-)-x < Z_\ell(t-) < \bar{Q}_\ell(t-) \text{ and } x \geq 0 \\ & \text{or } \bar{Q}_\ell(t-) < Z_\ell(t-) < Q_\ell(t-)-x \text{ and } x \leq 0. \end{cases}$$

Let

$$\bar{s}_\ell^{i'}(x) = \frac{s_\ell^{i'}(x)}{x} \qquad \text{for } x \neq 0 .$$

Then $\bar{s}_\ell^{i'}(x) \leq 1$ for $x \neq 0$.

For $t \in \mathcal{S}_k^1$,

$$\Delta|\bar{Q}_k(t) - Z_k(t)| = 0 ,$$

$$\Delta|\bar{Q}_j(t) - Z_j(t)| = s_j^{i'}(1-p_{kj}) ,$$

and

$$\Delta|\bar{Q}_\ell(t) - Z_\ell(t)| = s_\ell^{i'}(-p_{k\ell}) \quad \text{for } \ell \neq j,k.$$

We then have

$$E(\Delta|\bar{Q}_\ell(t) - Z_\ell(t)| \,|\, \bar{\mathcal{F}}_{i''-1},\ |\bar{Q}_{\varphi_k^i}(t-) - Z_{\varphi_k^i}(t-)| \geq 1) \leq 0 .$$

This is obvious when $|\bar{Q}_\ell(t-) - Z_\ell(t-)| \geq 1$ for $\ell \neq k$ since $P\{\varphi_k^i = j\} = p_{kj}$. Let $I_0 = \{j \neq k: |\bar{Q}_j(t-) - Z_j(t-)| < 1\}$, $\bar{p}_k = 1 - \sum_{j\in I_0} p_{kj}$ and

$$p'_{k\ell} = \begin{cases} \dfrac{p_{k\ell}}{\bar{p}_k} & \text{if } \bar{p}_k > 0 \\[2ex] 0 & \text{if } \bar{p}_k = 0 \end{cases}$$

for $\ell \notin I_0$. The result again follows by a simple calculation. So we have

$$E(H(t)|\mathfrak{F}_{i''-1}, |\bar{Q}_{\varphi_k^i}(t-) - Z_{\varphi_k^i}(t-)| \geq 1) \leq 0 .$$

For $t \in \mathcal{J}_k^2$,

$$\Delta|\bar{Q}_k(t) - Z_k(t)| = -1$$

and

$$\Delta|\bar{Q}_\ell(t) - Z_\ell(t)| = s_\ell^{i'}(-p_{k\ell}) \quad \text{for} \quad \ell \neq k.$$

Thus $H(t) \leq 0$, since $s_\ell^{i'}(-p_{k\ell}) \leq p_{k\ell}$ and $\sum_{\ell=1}^{K} p_{k\ell} \leq 1$.

For $t \in \mathcal{J}_k^3$, let $\delta = Z_k(t-)$, then

$$\Delta|\bar{Q}_k(t) - Z_k(t)| = -\delta ,$$

$$\Delta|\bar{Q}_j(t) - Z_j(t)| = s_j^{i'}(1 - [1-\delta]p_{kj}) ,$$

and

$$\Delta|\bar{Q}_\ell(t) - Z_\ell(t)| = s_\ell^{i'}(-[1-\delta]p_{k\ell}) \quad \text{for} \quad \ell \neq j,k.$$

Then

$$E(H(t)|\mathfrak{F}_{i''-1}, |\bar{Q}_{\varphi_k^i}(t-) - Z_{\varphi_k^i}(t-)| \geq 1) = -[1 - \sum_{\ell=1}^{K} \bar{s}_\ell^{i'} p_{k\ell}] \leq 0$$

by simple calculation. We will need a sharper result. If $p_{k0} > 0$ then

$$E(H(t)|\mathfrak{F}_{i''-1}, |\bar{Q}_{\varphi_k^i}(t-) - Z_{\varphi_k^i}(t-)| \geq 1) \leq -p_{k0}\delta .$$

regardless of the values of $\bar{s}_\ell^{i'}$ for $1 \leq \ell \leq K$. If $p_{k0} = 0$ and $\bar{s}_\ell^{i'} = 1$ for $\{\ell | p_{k\ell} > 0\}$ then $H(t) = 0$, while

$$E(H(t) | \mathcal{F}_{i''-i'}, |\bar{Q}_{\varphi_k^i}(t-) - Z_{\varphi_k^i}(t-)| \geq 1, \bar{s}_\ell^{i'} \neq 1 \text{ for some } \ell \neq k)$$

$$\leq 2 \min_{\{\ell | p_{k\ell} > 0\}} p_{k\ell}\delta .$$

For $t \in \mathcal{S}_k^4$, let $\delta = Z_k(t-)$, then

$$\Delta|\bar{Q}_k(t) - Z_k(t)| = -(1-\delta)$$

and

$$\Delta|\bar{Q}_\ell(t) - Z_\ell(t)| = s_\ell^{i''}(-(1-\delta)p_{k\ell}) \text{ for } \ell \neq k ,$$

so that $H(t) \leq 0$.

For $t \in \mathcal{S}_k^5$,

$$\Delta|\bar{Q}_k(t) - Z_k(t)| = -1,$$

$$\Delta|\bar{Q}_j(t) - Z_j(t)| = s_j^{i'}(1) ,$$

and

$$\Delta|\bar{Q}_\ell(t) - Z_\ell(t)| = 0 \text{ for } \ell \neq j, k,$$

so that $H(t) \leq 0$.

For $t \in \mathcal{S}_k^6$, both processes are on the interior and so $H(t) = 0$.

CHAPTER 5

THE LIMIT THEOREM

In this chapter we prove a limit theorem for a sequence of standard queueing networks, as defined in Section 4.2. The modified networks defined in Chapter 4 play an important role in the proof.

5.1. Statement of the Limit Theorem

We assume, as in Section 3.3, that we have an appropriately defined sequence of probability spaces $\{(\Omega^n, \mathfrak{F}^n, P^n), n \geq 1\}$. In the same way that we constructed $\underset{\sim}{Q}$, $\bar{\underset{\sim}{Q}}$, and $\underset{\sim}{Q}'$ on $(\Omega, \mathfrak{F}, P)$ in Chapter 4, we construct $\{\underset{\sim}{Q}^n, n \geq 1\}$, $\{\bar{\underset{\sim}{Q}}^n, n \geq 1\}$, and $\{\underset{\sim}{Q}'^n, n \geq 1\}$ on $\{(\Omega^n, \mathfrak{F}^n, P^n), n \geq 1\}$.

The heavy traffic theory, as explained in Chapter 1, deals with systems near saturation. Our definition of heavy traffic is essentially that used by Iglehart and Whitt (1970a,b) properly adapted to the more complicated setting of general networks. For our limit theorem we require that $\rho_i^n \to 1$ as $n \to \infty$ for each station i, and that furthermore $(\nu_i(n) - \mu_i(n))\sqrt{n} \to c_i$ as $n \to \infty$, with $-\infty < c_i < \infty$ for each i. This last requirement essentially says that the traffic intensities at the various stations approach unity at comparable rates. We may have $\rho_i^n \to 1$ from below for some stations i, $\rho_j^n \to 1$ from above for other stations j, and $\rho_k^n \equiv 1$ for still other stations k.

Let

$$(1) \qquad \underset{\sim}{Q}^{*n}(t) = n^{-1/2}\, \underset{\sim}{Q}^{n}(nt) \quad \text{for} \quad 0 \le t \le 1 \quad \text{and} \quad n \ge 1.$$

The following is the main result of this dissertation.

Theorem 3. Suppose

i) $\mu_k(n) \to \mu_k$ as $n \to \infty$ for $k = 1,\dots,K$,

ii) $\lambda_k(n) \to \lambda_k$ as $n \to \infty$ for $k = 1,\dots,K$,

iii) $a_k(n) \to a_k$ as $n \to \infty$ for $k = 1,\dots,K$,

iv) $s_k(n) \to s_k$ as $n \to \infty$ for $k = 1,\dots,K$,

v) $(\nu_k(n) - \mu_k(n))\sqrt{n} \to c_k$, $-\infty \le c_k < \infty$, for $k = 1,\dots,K$,

vi) $\max\limits_{1 \le k \le K} \sup\limits_{n \ge 1} E\{(v_k^1(n))^{2+\epsilon}\} < \infty$ for some $\epsilon > 0$,

vii) $\max\limits_{1 \le k \le K} \sup\limits_{n \ge 1} E\{(u_k^1(n))^{2+\epsilon}\} < \infty$ for some $\epsilon > 0$.

Then $\underset{\sim}{Q}^{*n} \Rightarrow \underset{\sim}{Z}^{*}$ in D^K as $n \to \infty$, where $\underset{\sim}{Z}^{*}$ is the process defined in Chapter 3.

Proof. The proof has several stages, which are taken up in the next two sections.

5.2. The Reflected Random Walk and the Borovkov Process

We first show that the normalized difference between the Borovkov process and the RRW converges weakly to zero. Let

$$\bar{Q}^{*n}(t) = n^{-1/2}\, \bar{Q}^n(nt) \quad \text{for} \quad 0 \le t \le 1 \quad \text{and} \quad n \ge 1. \tag{2}$$

Proposition 1. Under the hypotheses of Theorem 3, $\rho(\bar{Q}^{*n}, Z^{*n}) \Rightarrow 0$ as $n \to \infty$.

Proof. We first need to slightly modify the notation of Section 4.4. Let

$$T_k(n,t) = \{\sigma_k^n(\ell) \le t : Z_k^n(\sigma_k^n(\ell)-) = 0\}$$

for $n \ge 1$ and $t \ge 0$. Similarly define $T_k'(n,t)$, $T_k''(n,t)$, $\bar{T}_k(n,t)$, $\bar{T}_k(n,t)$, $\mathcal{S}(n,t)$, and $\mathcal{S}_k^i(n,t)$ for $1 \le i \le 6$. Let

$$D^n(t) = \sum_{k=1}^{K} |\bar{Q}^n(t) - Z^n(t)| \tag{3}$$

and $D^{*n}(t) = n^{-1/2}\, D^n(nt)$. By the triangle inequality it will suffice to show

$$\sup_{0 \le t \le 1} D^{*n}(t) \Rightarrow 0 \quad \text{as} \quad n \to \infty.$$

Using the definition of $H(t)$ from Section 4.4, we have

$$D^n(t) = \sum_{\mathcal{S}(n,t)} H(\sigma^n(\ell)) \,, \tag{4}$$

where for ease of notation we set

$$\sum_{\ell \ni \sigma^h(\ell) \in \mathcal{S}(n,t)} (\cdot) = \sum_{\mathcal{S}(n,t)} (\cdot) .$$

We use the same notational convention for other sets as well. The index of summation will always be obvious from the context.

By the discussion of Section 4.4,

$$D^n(t) = \sum_{i=1}^{6} \sum_{\mathcal{S}^i(n,t)} H(\sigma^n(\ell)) . \tag{5}$$

We can immediately dispense with $\mathcal{S}^2$, $\mathcal{S}^4$, $\mathcal{S}^5$ and $\mathcal{S}^6$ since for these sets $H(t) \leq 0$.

We deal first with $t \in \mathcal{S}^1_k$. Since every epoch in T_k results in an increase of one unit in Y_k, we have

$$|\mathcal{S}^1_k(n,t)| \leq |T_k(n,t)| \leq Y^n_k(t) , \tag{6}$$

where $|\cdot|$ applied to a set gives the cardinality. We denote the subsequence of $\mathcal{S}_k(n,t)$ in $\mathcal{S}^1_k(n,t)$ by ℓ_i, and the corresponding index in $\mathcal{S}$ by i'. Then

$$\sum_{\mathcal{S}^1_k(n,t)} H(\sigma_k(\ell_i)) = \sum_{\mathcal{S}^1_k(n,t)} [\sum_{j=1}^{K} s^{i'}_j (\phi^{\ell_i}_{kj} - p_{kj})]. \tag{7}$$

We need to further subdivide $\mathcal{S}^1_k(n,t)$. Let

$$\mathcal{S}_k^1(n,t) = \{\sigma_k(\ell_i) \in \mathcal{S}_k^2(n,t) : |\bar{Q}_{\varphi_k}^{\ell_i}(\sigma_k(i')-) - Y_{\varphi_k}^{\ell_i}(\sigma_k(i')-)| < 1$$

and $\bar{\mathcal{S}}_k^1(n,t) = \mathcal{S}_k^1(n,t) \setminus \bar{\mathcal{S}}_k^1(n,t)$. For $\sigma_k(\ell_i) \in \bar{\bar{\mathcal{S}}}_k^1$, the terms $s_j^{i'}(\Phi_{kj}^{\ell_i} - p_{kj})$ form a supermartingale difference sequence. From the proof of Proposition 3.1,

$$Y_k^n(t) \le [\underset{\sim}{M}^n(t)R]_k = \sum_{j=1}^{K} M_j^n(t) R_{jk} , \tag{8}$$

where

$$M_j^n(t) = - \inf_{0 \le s \le t} \{X_j^n(t)\} .$$

By Theorem 1, $\underset{\sim}{X}^{*n} \Rightarrow \underset{\sim}{X}^*$, so we can apply Proposition 2.5 to give

$$P\{ \sup_{0 \le t \le 1} \sum_{\bar{\bar{\mathcal{S}}}_k^1(n,nt)} H^n(t_i) > \sqrt{n}\,\epsilon\} \to 0 \quad \text{as} \quad n \to \infty, \tag{9}$$

for all $\epsilon > 0$. For $\sigma_k(\ell_i) \in \mathcal{S}_k^1(n,t)$ we know that

$$|\bar{Q}_{\varphi_k}^{\ell_i}(\sigma_k(i')) - Z_{\varphi_k}^{\ell_i}(\sigma_k(i'))| < 1$$

so that

$$\sup_{0 \le t \le 1} \sum_{\bar{\mathcal{S}}_k^1(n,nt)} H(t_i) < 2. \tag{10}$$

With the $\sqrt{n}$ normalization, this piece converges to zero.

The only part left to deal with is $\mathcal{S}^3$. Let

$$\mathcal{J}_k^3(n,t) = \{\sigma_k(\ell_i) \in \mathcal{S}_k^3(n,t) : |\bar{Q}_{\varphi_k}^{\ell_i}(\sigma_k(i')-) - Z_{\varphi_k}^{\ell_i}(\sigma_k(i')-)| < 1\}$$

and $\bar{\bar{\mathcal{J}}}_k^3(n,t) = \mathcal{S}_k^3(n,t) \setminus \mathcal{J}_k^3(n,t)$. We dispense with $\bar{\mathcal{J}}_k^3(n,t)$ just as with $\bar{\mathcal{S}}_k^1(n,t)$. We need to divide the relevant part of $T_k'(n,t)$ into two pieces. Let

$$\hat{T}_k'(n,t) = \{\sigma_k^n(\ell) \leq t \,|\, \delta^* < Z_k(\sigma_k^n(\ell)\ -) < 1\} \cap \bar{\bar{\mathcal{J}}}_k^3$$

and

$$\hat{\hat{T}}_k'(n,t) = \{\sigma_k^n(\ell) \leq t \,|\, 0 < Z_k|\sigma_k^n(\ell)-) \leq \ \delta^*\} \cap \ \bar{\bar{\mathcal{J}}}_k^3$$

for a fixed but arbitrary δ^* with $0 < \delta^* < 1$.

Since each epoch in T_k' results in an increase in Z_k of $(1-\delta)$, we have

$$\sum_{T_k'(n,t)} (1-\delta_i) \leq Y_k^n(t) ,$$

so that

$$(11) \qquad |\hat{\hat{T}}_k'(n,t)| \leq \frac{Y_k^n(t)}{1-\delta^*} .$$

We now let ℓ_i be the subsequence of $\hat{\hat{T}}_k'$, with corresponding index in $\mathcal{S}$ of i', and i" in $\mathcal{T}$. Then

$$E[-\delta_{i'} + \sum_{j=1}^{K} s_j^{i'}(\phi_{kj}^{\ell_i} - p_{kj}) \mid \mathcal{F}_{i''-1}] \le 0$$

from Section 4.4 and we can use Proposition 2.5 as above to get

(12) $$P\{\sup_{0 \le t \le 1} \sum_{\hat{T}_n'(n,nt)} H(t_i) > \sqrt{n}\,\epsilon\} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \epsilon > 0.$$

The only bound we have for $\hat{T}_k'(n,t)$ is

(13) $$|\hat{T}_k'(n,t)| \le S_k^n(t) .$$

Coupling the fact that $n^{-1}S_k^n(nt) \xrightarrow{a.s.} \mu_k t$ as $n \to \infty$ for $1 \le k \le K$ and $0 \le t \le 1$ with

(14) $$E[-\delta_{j'} + \sum_{j=1}^{K} s_j^{i'}(\phi_{kj}^{\ell_i} - p_{kj}) \mid \mathcal{F}_{i-1}, \bar{s}_\ell^{i'} \ne 1 \text{ for some } \ell \ne k]$$

$$\le -2\delta^* \min\{p_{kj} \mid 1 \le j \le K \text{ and } p_{kj} > 0\} ,$$

we can use Proposition 2.6 to get

(15) $$P\{\sup_{0 \le t \le 1} \sum_{\hat{T}_k'(n,nt)} H(t_i) > \sqrt{n}\,\epsilon\} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \epsilon > 0.$$

Combining (9), (10), (12) and (15) yields the result.

5.3. The Borovkov Process and the Standard Network

In order to show that the normalized difference between the standard and Borovkov networks converge weakly to zero, we use the I/O network as an intermediate step. We start with the easier part which deals with the standard network and I/O network. Let

$$\underset{\sim}{Q}'^{\#n}(t) = n^{-1/2}\ \underset{\sim}{Q}'^{n}(nt) \quad \text{for} \quad 0 \le t \le 1 \quad \text{and} \quad n \ge 1\ .$$

Proposition 2. Under the hypotheses of Theorem 3, $\rho(\underset{\sim}{Q}^{\#n}, \underset{\sim}{Q}'^{\#n}) \Rightarrow 0$ as $n \to \infty$.

Proof. By construction, each station in the I/O network has the same input as the identically indexed station in the standard network. Thus we can use Theorem 3.1 of Iglehart and Whitt (1970a) which says that for single station systems, the normalized difference between the Borovkov and standard systems converges weakly to zero. This gives us $\rho(Q_k^{\#n}, Q_k'^{\#n}) \Rightarrow 0$ as $n \to \infty$ for $1 \le k \le K$. The triangle inequality yields the result.

Before proving the other part, involving the I/O network and Borovkov network, we present two lemmas which are needed in the proof. Let

$$(12) \qquad N_\ell^{\#n}(t) = n^{-1/2}[N_\ell^n(nt) - \gamma_\ell(n)nt] \quad \text{for} \quad 1 \le \ell \le K,\ 0 \le t \le 1 \quad \text{and} \quad n \ge 1.$$

Lemma 1. Under the hypotheses of Theorem 3, $N_\ell^n \Rightarrow \psi_\ell$ for $1 \le \ell \le K$, where ψ_ℓ is Brownian Motion with zero drift and variance $\Lambda_\ell = \sum_{k=1}^{K} \lambda_k R_{k\ell}(1 + \sqrt{\lambda_k a_k})$.

Proof. Let $N_\ell^n(t,k) = \sum_{j=1}^{A_k^n(t)} m(j,k,\ell)$ for $1 \le \ell,\ k \le K,\ t \ge 0$, and $n \ge 1$,

$$N_\ell^{*n}(t,k) = n^{-1/2}[N_\ell^n(nt,k) - \lambda_k(n)R_{k\ell}nt] \quad \text{for}$$

for $1 \le \ell,\ k \le K,\ 0 \le t \le 1$, and $n \ge 1$,

$$\bar{N}_\ell^{*n}(t,k) = n^{-1/2} \sum_{j=1}^{[\lambda_k nt]} (m(j,k,\ell) - R_{k\ell})$$

for $1 \le \ell,\ k \le K,\ 0 \le t \le 1$, and $n \ge 1$,

and

$$\bar{\bar{N}}_\ell^{*n}(t,k) = n^{-1/2} \sum_{j=1}^{A_k^n(nt)} (m(j,k,\ell) - R_{k\ell})$$

for $1 \le \ell,\ k \le K,\ 0 \le t \le 1$, and $n \ge 1$.

In order to get $m(j,k,\ell)$ to be independent of n, we work with a distributionally equivalent standard network where each customer carries his own sequence of routing decisions. Thus the N_ℓ^n we deal with here has the same measure as that of the standard network defined in Section 4.2.

By Proposition 2.3,

$$\bar{\underset{\sim}{N}}^{*n}(\cdot,k) \Rightarrow \eta_{\ell k}(\cdot) \quad \text{in } D \text{ for } 1 \leq \ell,\ k \leq K,$$

where $(\eta_{\ell 1},\ldots,\eta_{\ell k})$ are independent Brownian motions with zero drift and variances $\lambda_k R_{k\ell}$. (When $P_{k\ell} = 0$, $\eta_{\ell k}$ is identically zero.)
We have

$$N_\ell^{*n}(t,k) = \bar{\bar{N}}_\ell^{*n}(t,k) + R_{k\ell} A_k^{*n}(t) \ . \tag{13}$$

Let $\hat{N}_\ell^{*n}(t,k) = \bar{N}_\ell^{*n}(t,k) + R_{k\ell} A_k^{*n}(t)$, then

$$\hat{N}_\ell^{*n}(\cdot,k) \Rightarrow \eta_{\ell k}(\cdot) + R_{k\ell}\sqrt{\lambda_k^3 a_k}\ \beta_k(\cdot) \quad \text{in } D \text{ for } 1 \leq \ell,\ k \leq K.$$

By the same argument used to show $\rho(\underset{\sim}{S}^{*n}, \bar{\underset{\sim}{S}}_k^{*n}) \Rightarrow 0$ in Theorem 1, we get $\rho(\bar{\bar{N}}_\ell^{*n}(\cdot,k), \bar{N}_\ell^{*n}(\cdot,k)) \Rightarrow 0$ for $1 \leq \ell,\ k \leq K$. By the converging together theorem,

$$N_\ell^{*n}(\cdot,k) \Rightarrow \eta_{\ell k} + R_{k\ell}\sqrt{\lambda_k^3 a_k}\ \beta_k(\cdot) \quad \text{in } D \text{ for } 1 \leq \ell,\ k \leq K.$$

By independence of the components,

$$N_\ell^{*n}(\cdot) = \sum_{k=1}^{k} N_\ell^{*n}(\cdot,k) \rightarrow \sum_{k=1}^{k} (\eta_{\ell k}(\cdot) + R_{k\ell}\sqrt{\lambda_k^3 a_k}\ \beta_k(\cdot)) \overset{d}{=} \psi_\ell(\cdot). \tag{14}$$

This completes the proof.

Let

$$\hat{S}_k^{*n}(t) = n^{-1/2}[\hat{S}_k^n(nt) - \mu_k(n)nt] \qquad \text{for } 1 \le k \le K \text{ and } 0 \le t \le 1,$$

$$\hat{S}_{kj}^{*n}(t) = n^{-1/2}[\hat{S}_{kj}^n(nt) - p_{kj}\mu_k(n)nt] \qquad \text{for } 1 \le j,k \le K \text{ and } 0 \le t \le 1,$$

and

$$\hat{\underline{S}}_k^{*n} = (\hat{S}_k^{*n}, \hat{S}_{k1}^{*n}, \dots, \hat{S}_{kK}^{*n}) .$$

Lemma 2. The sequence $\{\hat{\underline{S}}_k^{*n}, n \ge 1\}$ is tight for $1 \le k \le K$.

Remark. Tightness is a condition on a sequence of stochastic processes which guarantees that every subsequence has a further subsequence with a weak limit. Conversely, any sequence of stochastic processes which converges weakly is tight. (See Billingsley (1968), Theorems 6.1 and 6.2.)

Proof. By the definition of tightness, we need to show that for all ϵ, $\eta > 0$, there exists a $\delta \in (0,1)$ and an $n_0 > 0$ such that for $n > n_0$,

$$P\{\sup_{|u-t|<\delta} (|\hat{S}_k^{*n}(u) - \hat{S}_k^{*n}(t)| + \sum_{j=1}^{k} |\hat{S}_{kj}^{*n}(u) - \hat{S}_{kj}^{*n}(t)|) \ge \epsilon\} \le \eta .$$

Without loss of generality, we will henceforth assume that $u > t$. We know that $\{S_k^{*n}, n \ge 1\}$ is tight, since it converges weakly. By definition of $\hat{S}_k$, we have

$$\text{(15)} \qquad \hat{S}_k^{*n}(nu) - \hat{S}_k^n(nt) \le S_k^n(nu + \theta_k^n(nu)) - S_k^n(nt + \theta_k^n(nt)),$$

where θ_k^n is the shift operator which lines up the service process for the kth station of the standard system with S_k^n. (See Iglehart and Whitt (1970a), Theorem 3.1 for a complete description of the shift.) From Iglehart and Whitt (1970a), we know that

$$(16)\quad P\{\sup_{0\le u,t\le 1} |S_k^n(nu)-S_k^n(nt)-S_k^n(nu+\theta_k^n(nu))+S_t^n(nt+\theta_k^n(nt))| \ge \frac{\sqrt{n}\,\epsilon}{4}\} \le \eta .$$

Thus

$$(17)\quad P\{\sup_{|u-t|<\delta} \hat{S}_1^n(nu) - \hat{S}_k^n(nt) - \mu_k(n)\, n(u-t) > \frac{\sqrt{n}\,\epsilon}{2}\} \le \eta .$$

To get the bound on the other side, we use the load process $\underset{\sim}{N}$. We have

$$(18)\quad \hat{S}_k^n(nu) - \hat{S}_k^n(nt) = [N_k^n(nu) - N_k^n(nt)] - [L_k^n(nu) - L_k^n(nt)] .$$

By Lemma 1, $\{N_k{}^n, n \ge 1\}$ is tight, so

$$(19)\quad P\{\sup_{|u-t|<\delta} |N_k^n(nu) - N_k^n(nt) - \gamma_k^n n(u-t)| > \frac{\sqrt{n}\,\epsilon}{8}\} \le \eta$$

and

$$(20)\quad P\{\sup_{|u-t|<\delta} |N_k{}^n(nu) - N_k{}^n(nt) - \mu_k{}^n\, n(u-t)| > \frac{\sqrt{n}\,\epsilon}{4}\} \le \eta$$

since $(\gamma_k^n - \mu_k^n)\sqrt{n} \to (cR)_k$. We want to show that $P\{\sup L_k^n(nu) - L_k^n(nt) > \sqrt{n}\,\epsilon/4\} \le \eta$. This is done via the queue length process. We claim that for any ϵ', $\eta' > 0$ there exists a $\delta_k' > 0$ and $n_k > 0$ such that for $n > n_k$,

$$(21) \qquad P\{\sup_{|u-t|<\delta} Q_k^n(nu) - Q_k^n(nt) > \sqrt{n}\,\epsilon'\} \le \eta' .$$

Since $Q_k^n(\cdot) \ge 0$, we can assure without loss of generality that $Q_k^n(nt) = 0$ and that $Q_k^n(ns) > 0$ for $t < s \le u$. We then have

$$Q_k^n(nu) \le [A_k^n(nu)-A_k^n(nt)] + \sum_{j=1}^{k} [S_{jk}^n(nu + \theta_j^n(nu)) - S_{jk}^n(nt + \theta_j^n(nt))]$$

$$- [S_k^n(nu + \theta_k^n(nu)) - S_k^n(nu + \theta_k^n(nt))] .$$

The tightness of $\{A_k^{*n}, n \ge 1\}$, $\{S_k^{*n}, n \ge 1\}$ and $\{S^{*n}, n \ge 1\}$ give the result (21).

From equation (4.8) we know that

$$\max_{1\le \ell \le K} [K \max_{1\le j \le K} R_{j\ell}]^{-1} E[L_\ell^n(t) | \mathfrak{F}_t^n] \le \max_{1 < j \le K} Q_j^n(t) .$$

Let $\delta_0' = \min_{1\le k\le K} \delta_k'$ and $n_0'' = \max n_k$, then

$$P\{\sup_{|u-t|<\delta_0'} E[L_\ell^n(nu) | \mathfrak{F}_{nu}^n] - E[L_\ell^n(nt) | \mathfrak{F}_{nt}^n] \ge \sqrt{n}\,\epsilon'\} \ge \eta'$$

Let $\eta' = \eta/2$. We work on the set

$$\{\sup_{|u-t|<\delta_0'} E[L_\ell^n(nu) | \mathfrak{F}_{nu}^n] - E[L_\ell^n(nt) | \mathfrak{F}_{nt}^n] < \sqrt{n}\,\epsilon'\} ,$$

on which

$$P\{\sup_{|u-t|<\delta} L_\ell^n(nu) - L_\ell^n(nt) \geq \sqrt{n}\,\epsilon/4\} \leq 4\epsilon'/\epsilon + \eta/2$$

by a simple Chebychev type inequality. Choosing $\epsilon' = \eta\epsilon/8$, gives the $|\hat{S}_k^{*n}(u) - \hat{S}_k^{*n}(t)|$ part.

To show $P\{\sup_{|u-t|<\delta} |\hat{S}_{kj}^{*n}(u) - \hat{S}_{kj}^{*n}(t)| \geq \epsilon/2K\} \leq \eta$,

we work on the set

$$\{\sup_{|u-t|<\delta} |\hat{S}_k^{*n}(u) - \hat{S}_k^{*n}(t)| < \epsilon''\},$$

which has probability measure at least $(1 - \eta/2)$. Using Kolmogorov's inequality,

$$P\{\sup_{|u-t|<\delta} \sum_{[\mu_j(n)nt]}^{[\mu_j(u)nu+\sqrt{n}\,\epsilon'']} (\Phi_{jk}^{\ell_i} - p_{jk})| \geq \sqrt{n}\,\epsilon/2k\}$$

$$\leq [n\epsilon^2]^{-1}\, 4k^2 p_{jk}(1-p_{jk})[\mu_j(n)n\delta + \sqrt{n}\,\epsilon''] + \frac{\eta}{2}.$$

We can clearly choose n_1 (so $n \geq n_1$) and δ to yield the result.

<u>Proposition 6</u>. Under the hypotheses of Theorem 3, $\rho(Q'^{*n}, Q^{*n}) \Rightarrow 0$ as $n \to \infty$.

<u>Proof</u>. Let

$$S_k'^{*n}(t) = n^{-1/2}[S_k'^n(nt) - \mu_k(n)nt] \quad \text{for } 1 \leq k \leq K,\ n \geq 1 \text{ and } 0 \leq t \leq 1,$$

and

$$A_k'^{*n}(t) = n^{-1/2}[A_k'^n(nt) - \nu_k(n)nt] \quad \text{for } 1 \leq k \leq K,\ n \geq 1 \text{ and } 0 \leq t \leq 1.$$

Then we have

$$A'^{\#n}_k(t) = A^{\#n}_k(t) + \sum_{j=1}^{k} \hat{S}^{\#n}_{jk}(t)$$

and

$$S'^{\#n}_k(t) = S^{\#n}_k(t) + \inf_{0 \leq s \leq t} \{A'^{\#n}_k(s) - S^{\#n}_k(s) + c_k^n s\}.$$

Just as in Section 4.3, we can write

$$\underset{\rightarrow}{S}'^{\#n} = h_n(\underset{\sim}{A}^{\#n}, \underset{\rightarrow}{S}^{\#n}, \underset{\rightarrow}{S}^{\#n}, \underset{\rightarrow}{\varphi})$$

and

$$\underset{\rightarrow}{\bar{S}}^{\#n} = h_n(\underset{\sim}{A}^{\#n}, \underset{\rightarrow}{S}^{\#n}, \underset{\rightarrow}{\bar{S}}^{\#n}, \underset{\rightarrow}{\varphi}) ,$$

with $\underset{\rightarrow}{\bar{S}}^{\#n}$ uniquely defined subject to $\underset{\rightarrow}{\bar{S}}^{\#n}(0) = 0$. From Lemma 4.1 of Iglehart and Whitt (1970a) we have $\rho(S_k^{\ n}, S'^{\#n}_k) \Rightarrow 0$ as $n \to \infty$ for $1 \leq k \leq K$. The inequality

$$|\hat{S}^{\#n}_k(t) - S'^{\#n}_{jk}(t)| \leq |\hat{S}^{\#n}_k(t) - S'^{\#n}_k(t)|$$

gives $\rho(\hat{S}^{\#n}_{kj}, S'^{\#n}_{kj}) \Rightarrow 0$ as $n \to \infty$ for $1 \leq j,\ k \leq K$. So $\rho(\underset{\rightarrow}{\hat{S}}^{\#n}, h_n(\underset{\sim}{A}^{\#n}, \underset{\rightarrow}{S}^{\#n}, \underset{\rightarrow}{\hat{S}}^{\#n}, \underset{\rightarrow}{\varphi})) \Rightarrow 0$ as $n \to \infty$.

By the tightness of $\{\underset{\rightarrow}{\hat{S}}^{\#n}, h \geq 1\}$ we can take a subsequence n' such that $\underset{\rightarrow}{\hat{S}}^{\#n'} \Rightarrow \underset{\rightarrow}{\hat{S}}^{*}$ and hence $\underset{\rightarrow}{S}'^{\#n} \Rightarrow \underset{\rightarrow}{\hat{S}}^{*}$. By the continuous mapping theorem,

$$h_n(A^{\#n}, S^{\#n}, \hat{S}^{\#n}, \varphi) \Rightarrow h_0(A^*, \underset{\rightarrow}{S}^*, \underset{\rightarrow}{\hat{S}}^*, \underset{\rightarrow}{\varphi})$$

where h_0 is the same as h_n except that c_k^n is replaced by c_k. Thus

$$\hat{\underset{\rightarrow}{S}}^* = \bar{\underset{\rightarrow}{S}}^* ,$$

where $\bar{\underset{\rightarrow}{S}}^*$ is the unique solution to

$$\bar{\underset{\rightarrow}{S}}^* = h_0(\underset{\sim}{A}^*, \underset{\rightarrow}{S}^*, \bar{\underset{\rightarrow}{S}}^*, \underset{\rightarrow}{\varphi})$$

subject to $\bar{\underset{\rightarrow}{S}}^*(0) = 0$, which we know to exist since $\underset{\sim}{Q}^{\#n} \Rightarrow Z^*$.

Every convergent subsequence of $\{\hat{\underset{\rightarrow}{S}}^{\#n}, n \geq 1\}$ has the limit $\bar{S}^*$ by the argument above, so the limit is unique. Since

$$Q_k'^{\#n}(t) = A_k'^{\#n}(t) - S_k'^{\#n}(t) + c_k^n t$$

is a continuous function of its data,

$$\rho(\underset{\sim}{Q}'^{\#n}, \bar{\underset{\sim}{Q}}^{\#n}) \Rightarrow 0 \quad \text{as} \quad n \to \infty .$$

Combining Propositions 4, 5 and 6 we get $\rho(Q^{\#n}, Y^{\#n}) \Rightarrow 0$ as $n \to \infty$, so that Theorem 3 follows from Theorem 2 and an application of the converging together theorem.

CHAPTER 6

PROPERTIES OF THE LIMIT PROCESS

This chapter is devoted to a discussion of the limit process $\underset{\sim}{Z}^*$. We begin by relating the definition of Z^* given in section 1.4 to the analytic theory of Markov processes. In the second section, we derive a partial differential equation which must be satisfied by the stationary distribution (if one exists). In Section 3 we present a family of special cases for which we have solved the partial differential equation, and in Section 4 we show that Kobayashi's stationary distribution solves the partial differential equation for a subset of these cases.

6.1. The Limit Process is a Diffusion

Multidimensional diffusions with boundaries have previously been studied by Bellot (1975), Stroock and Varadhan (1971), Watanabe (1971), and several other authors. Their work does not cover our limit diffusion $\underset{\sim}{Z}^*$. Bellot (1975) considers only the case of normal reflection in two dimensional regions, while Stroock and Varadhan (1971) and Watanabe (1971) impose smoothness restrictions on the boundaries which are not satisfied in our case. The following definition of a diffusion is broad enough to cover our process, however. It is due to Ito and McKean (1965).

Definition. A stochastic process $W = \{W(t),\ t \geq 0\}$ is a diffusion if

i) it satisfies $P\{W(T+s) \in A \mid \mathcal{F}_t\} = P\{W(T+s) \in A \mid W(T)\}$ for $A \in \mathcal{F}_{T+s}$, where $\mathcal{F}_t$ is the σ-field generated by $\{W(s),\ 0 \leq s \leq t\}$, T is any stopping time, and $s \geq 0$, and

ii) $R_\alpha : g(\cdot) \to E_\cdot[\int_0^\infty e^{-\alpha t} g(W(t))dt]$ maps bounded continuous functions into bounded continuous functions.

Condition (i) is the strong Markov property and (ii) is a kind of regularity which implies continuity of paths.

Remark. If $\underset{\sim}{X}^*(0) \in \mathbb{R}_+^K$, then $\underset{\sim}{Z}^*(0) = \underset{\sim}{X}^*(0)$. Thus the notation $P_x(\cdot)$ can be interpreted as the probability measure on the path space of Z^* *corresponding either to* $\underset{\sim}{X}^*(0) = x$ or $\underset{\sim}{Z}^*(0) = x$, when $x \in \mathbb{R}_+^K$. To have $P_x(\cdot)$ make sense we need to show that $\underset{\sim}{Z}^*$ is a Markov process. This will be a result of i), so that after it is proven we are justified in our use of this notation.

Proposition 1. $\underset{\sim}{Z}^*$ is a diffusion.

Proof. We verify $(\cdot)$ using the relation between $\underset{\sim}{Z}^*$ and $\underset{\sim}{X}^*$, and the strong Markov property for $\underset{\sim}{X}^*$. Note that (i) is equivalent to

$$P\{\underset{\sim}{Z}^*(T+s) - \underset{\sim}{Z}^*(T) \mid \mathcal{F}_T\} = P\{\underset{\sim}{Z}^*(T+s) - \underset{\sim}{Z}^*(T) \mid \underset{\sim}{Z}^*(T)\} .$$

From the definition of $\underset{\sim}{Z}^*$ we have

$$\underset{\sim}{Z}^*(T+s) - \underset{\sim}{Z}^*(T) = [\underset{\sim}{X}^*(T+s) - \underset{\sim}{X}^*(T)] + [(I-P^T)[\underset{\sim}{Y}(T+s) - \underset{\sim}{Y}(T)].$$

The term $[\underset{\sim}{X}^*(T+s) - \underset{\sim}{X}^*(T)]$ is independent of $\mathcal{F}_T$ by the strong Markov property and independent increments. The term $[\underset{\sim}{Y}(T+s) - \underset{\sim}{Y}(T)]$ is independent of the past conditioned on $\underset{\sim}{Z}^*(T)$ by the definition of $\underset{\sim}{Y}$.

To verify (ii) we again use the relation between $\underset{\sim}{Z}^*$ and $\underset{\sim}{X}^*$. Let $g: \mathbb{R}_+^K \to \mathbb{R}$ be bounded and continuous, and let

$$h(x) = R_\alpha g(x) = E_x[\int_0^\infty e^{-\alpha t} g(\underset{\sim}{Z}^*(t))dt]$$

for $x \in \mathbb{R}_+^K$. For purposes of this proof only, we introduce the notation $\underset{\sim}{X}_x^*$ and $\underset{\sim}{Z}_x^*$ to correspond to the processes starting at $x \in \mathbb{R}_+^K$. Since a shift in the starting state of $\underset{\sim}{X}^*$ shifts the entire path of the process by the same amount, if $|x-x'| < \delta$, then

$$\sup_{0 \le t < \infty} |\underset{\sim}{X}_x^*(t) - \underset{\sim}{X}_{x'}^*(t)| < \delta \ .$$

By the continuity of $f: C^K \to C^K$ which defines $\underset{\sim}{Z}^*$ in terms of $\underset{\sim}{X}^*$, for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x-x'| < \delta$, then

$$\sup_{0 \le t < \infty} |\underset{\sim}{Z}_x^*(t) - \underset{\sim}{Z}_{x'}^*(t)| < \epsilon \ .$$

The continuity of g implies that for any $\eta > 0$, there exists an $\epsilon > 0$ such that if $|x-x'| < \delta(\epsilon)$, then

$$\sup_{0 \le t < \infty} |g(\underset{\sim}{Z}_x^*(t)) - g(\underset{\sim}{Z}_{x'}^*(t))| < \eta \ .$$

Thus $|h(x) - h(x')| < \eta/\alpha$ and so h is continuous. The bounded part is obvious from boundedness of g.

In order to describe $\underset{\sim}{Z}^*$ in terms of its generator we need to introduce some notions from the analytic theory of Markov processes. In defining the weak infinitesimal generator we follow Dynkin (1965) specializing his treatment for general state spaces to $\mathbb{R}_+^K$. Most of the information pertaining to Markov processes in Euclidean state spaces is also contained in Breiman (1968). Let B be the set of all bounded and measurable functions from $\mathbb{R}_+^K$ into $\mathbb{R}$, and define

$$T_t g(\alpha) = E_x[g(\underset{\sim}{Z}^*(t))] \quad \text{for } g \in B,\ x \in \mathbb{R}_+^K \text{ and } t \geq 0 .$$

Also, let B_0 be the subset of B for which $T_t g \to g$ as $t \downarrow 0$, and let $\mathcal{D}$ be the subset of B_0 for which $(T_t g - g)/t$ converges boundedly pointwise to a limit in B_0. We denote the limit by $\mathcal{G}g$, and the operator $\mathcal{G}$ is called the (weak infinitesimal) generator of $\underset{\sim}{Z}^*$. The set $\mathcal{D}$ is called the domain of the generator.

<u>Remark</u>. Bounded pointwise convergence is pointwise convergence with the added condition that

$$t^{-1} \sup_{x \in \mathbb{R}_+^K} |(T_t g)(x) - g(x)| \leq M < \infty \quad \text{for all } t \text{ small enough.}$$

(See Breiman (1968), p. 340.) The type of convergence used by Dynkin (1965) is weak convergence for linear functionals on the Banach space consisting of all real, bounded, measurable functions on the state space. When the state space is Euclidean, this specializes to bounded pointwise convergence.

We say that $g: \mathbb{R}_+^K \to \mathbb{R}$ vanishes in some neighborhood of every corner if there exists an $\epsilon > 0$ such that if $x_i, x_j \geq 0$ with $1 \leq i < j \leq K$ and $x_i + x_j \leq \epsilon$ then $g(x) = 0$. We let $\mathcal{C}^0$ denote the set of all $g: \mathbb{R}_+^K \to \mathbb{R}$ which vanish in some neighborhood of every corner and have bounded derivatives of all orders. We introduce the notational convention $x^i = (x_1, \ldots, 0, \ldots, x_K)$, where the ith component of $x \in \mathbb{R}^K$ is replaced by zero.

Proposition 2. Suppose $h \in \mathcal{C}^0$. Then $h \in \mathcal{D}$ if and only if

$$(1) \qquad \nabla h(x^i) \cdot (e_i - p_{i.}) = 0 \quad \text{for } 1 \leq i \leq K,\ x \in \mathbb{R}_+^K ,$$

in which case

$$(2) \qquad \mathcal{J} h(x) = \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_{ij}^2 \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{k} c_i \frac{\partial h}{\partial x_i}(x) \quad \text{for } x \in \mathbb{R}_+^K .$$

Proof. We define $o(t)$ as usual, so that if $g(x,t)$ is $o(t)$, then $g(x,t)/t \to 0$ as $t \downarrow 0$ for all $x \in \mathbb{R}_+^K$. We say $g(x,t)$ is $\hat{o}(t)$ if $g(x,t)/t \to 0$ boundedly pointwise as $t \downarrow 0$ for $x \in \mathbb{R}_+^K$.
By Taylor's theorem,

$$(3) \qquad h(x + w) = h(x) + wh'(x) + \frac{1}{2} w'h''(x)w + Cw^* ,$$

where

$$C = \sup_{x \in \mathbb{R}_+^K} |h'''(x)| < \infty \quad \text{and} \quad |w^*| \leq |w| .$$

We begin by considering x lying on one boundary, so that $x_i = 0$, $x_j > 0$ for $j \neq i$. Until $\underset{\sim}{Z}^*$ hits another boundary, $Y_i(t) = M_i(t) = - \inf_{0 \leq s \leq t} \{X_i^*(s)\}$, and $Y_j(t) = 0$ for $j \neq i$. Let $\hat{\underset{\sim}{Z}} = \{\hat{\underset{\sim}{Z}}(t),\ t \geq 0\}$ be the process defined by

$$\hat{Z}_i(t) = X_i^* + M_i(t)$$

(4) and

$$\hat{Z}_j(t) = X_j^*(t) - p_{ij} M_i(t) + x_j \qquad \text{for } j \neq i.$$

Let

$$q_{i\ell} = \begin{cases} 1 & \ell = i \\ -p_{i\ell} & \ell \neq i \end{cases}$$

then $\hat{Z}_\ell(t) = X_\ell^*(t) + q_{i\ell} M_i(t)$ for $1 \leq \ell \leq K$. Using the fact that h vanishes in every corner, as well as the first exit distribution for multidimensional Brownian motion (Ito and McKean (1965), p. 234) we have $|E_x h(\underset{\sim}{Z}^*(t)) - E_x h(\hat{\underset{\sim}{Z}}(t))| = \hat{o}(t)$. So we compute the generator for $\hat{\underset{\sim}{Z}}$. We have

(5) $(T_t h)(x) - h(x)$

$$= \sum_{\ell=1}^{K} h_\ell(x)\, E_x[\Delta\hat{Z}_\ell(t)] + \sum_{\ell=1}^{K} \sum_{k=1}^{K} h_{\ell k}(x)\, E_x[\Delta\hat{Z}_\ell(t)\Delta\hat{Z}_k(t)] + \hat{o}(t),$$

where $\Delta\hat{Z}_\ell(t) = \hat{Z}_\ell(t) - \hat{Z}_\ell(0)$, and the $\hat{o}(t)$ is bounded using bounds on the third moment of Brownian motion. Let

$$\hat{M}_i(t) = \sup_{0 \leq s \leq t} \{X_i^*(t)\} .$$

From known results on Brownian motion,

(6) $$t^{-1} E_x[M_i(t)] \to \infty \quad \text{as} \quad t \downarrow 0 \quad \text{and} \quad t^{-1} E_x[\hat{M}_i(t)] \to \infty \quad \text{as} \quad t \downarrow 0$$

(7) $$t^{-1} E_x[M_i^2(t)] \to \sigma_i^2 \text{ as } t \downarrow 0 \quad \text{and} \quad t^{-1} E_x[\hat{M}_i^2(t)] \to \sigma_i^2 \text{ as } t \downarrow 0,$$

(8) $$t^{-1} E_x[X_i^*(t)] \to c_i \text{ as } t \downarrow 0 \quad \text{and} \quad t^{-1} E_x[X_i^*(t)\ X_j^*(t)] \to \sigma_{ij}^2 \text{ as } t \downarrow 0.$$

Using (5), the definition of $\hat{\underset{\sim}{Z}}$ and (6)-(8),

(9) $$(T_t h)(x) - h(x)$$
$$= \sum_{\ell=1}^{K} h_\ell(x)\ c_\ell t + E[M_i(t)] \sum_{\ell=1}^{K} h_\ell(x)\ q_{i\ell}$$
$$+ \sum_{\ell=1}^{K} \sum_{k=1}^{K} h_{\ell k}(x)\ E_x[(X_\ell^*(t) + q_{i\ell}M_i(t))(X_k^*(t) + q_{ik}M_i(t))] + \delta(t).$$

We have

(10) $$E_x[(X_\ell^*(t) + q_{i\ell}M_i(t))(X_k^*(t) + q_{ik}M_i(t))]$$
$$= E_x[X_\ell^*(t)X_k^*(t)] + q_{ik}E_x[X_\ell^*(t)M_i(t)] + q_{i\ell}E_x[X_k^*(t)M_i(t)]$$
$$+ q_{i\ell}q_{ik}\ E_x[M_i^2(t)] .$$

It is known that $X_i^*(t) + M_i(t)$ has the same distribution as $\hat{M}_i(t)$ for all $t > 0$ (see the proof of Proposition 3 in Harrison (1977)). Thus

$$E_x[(X_i^*(t))^2 + 2X_i^*(t)M_i(t) + M_i^2(t)] = E_x[\hat{M}_i^2(t)] \tag{11}$$

and

$$E_x[X_j^*(t)M_i(t)] = E_x[X_j^*(t)(\hat{M}_i(t) - X_i^*(t))] \quad \text{for } j \neq i. \tag{12}$$

Using (8) and (11),

$$t^{-1} E_x[X_i^*(t)\, M_i(t)] \to -\frac{1}{2}\sigma_{ii}^2 \quad \text{as } t \downarrow 0. \tag{13}$$

By symmetry,

$$\begin{aligned} &E_x[(X_j^*(t) - c_j t) \inf_{0 \le s \le t} \{X_i^*(x) - c_i s\}] \\ &\quad = E_x[(X_j^*(t) - c_j t) \sup_{0 \le s \le t} \{X_i^*(s) - c_i s\}] . \end{aligned} \tag{14}$$

We have

$$\sup_{0 \le s \le t} \{X_i^*(s) - c_i s\} - |c_i|t \le \sup_{0 \le s \le t} \{X_i^*(s)\} \le \sup_{0 \le s \le t} \{X_i^*(s) - c_i s\} + |c_i|t$$

and

$$\inf_{0 \le s \le t} \{X_i^*(s) - c_i s\} - |c_i|t \le \inf_{0 \le s \le t} \{X_i^*(s)\} \le \inf_{0 \le s \le t} \{X_i^*(s) - c_i s\} + |c_i|t,$$

which gives

$$\hat{M}_i(t) = \sup_{0 \le s \le t} \{X_i^*(s) - c_i s\} + c_i s'$$

and

$$M_i(t) = - \inf_{0 \le s \le t} \{X_i^*(s) - c_i s\} - c_i s''$$

for some s', s'' with $-t \le s', s'' \le t$. Substituting this into (14)

$$E_x[-(X_j^*(t)-c_j t)(M_i(t)+c_i s'')] = E_x[(X_j^*(t)-c_j t)(\hat{M}_i(t)-c_i s')]$$

or

$$(15) \quad E_x[X_j^*(t)M_i(t)] + E_x[X_j^*(t)\hat{M}_i(t)]$$

$$= c_j t\ E_x[M_i(t)] + c_i c_j t s'' - c_i s''\ E_x[X_j^*(t)]$$

$$+ c_j t\ E_x[\hat{M}_i(t)] - c_i c_j t s' - c_i s'\ E_x[X^*(t)] .$$

Combining (12) and (15),

$$(16) \qquad E_x[X_j^*(t)\ M_i(t)] = -\frac{1}{2} E_x[X_j^*(t)\ X_i(t)] + \hat{o}(t) .$$

Substituting (13) and (16) into (10),

$$(17) \quad t^{-1} E_x[(X_\ell^*(t) + q_{i\ell} M_i(t))(X_k(t) + q_{ik} M_i(t))]$$

$$\longrightarrow \sigma_{\ell k}^2 + \frac{1}{2} q_{ik}\sigma_{\ell i}^2 + \frac{1}{2} q_{i\ell}\sigma_{ki}^2 + q_{i\ell} q_{ik} \sigma_{ii}^2 .$$

Thus for the second order terms we have

$$(18)\quad \sum_{\ell=1}^{K}\sum_{k=1}^{K} h_{\ell k}(x)\, E_x[(X_\ell^*(t) + q_{i\ell}M_i(t))(X_k^*(t) + q_{ik}M_i(t))]$$

$$= \sum_{\ell=1}^{K}\sum_{k=1}^{K} h_{\ell k}(x)\, \sigma_{\ell k}^2 + \sum_{k=1}^{K} (\sigma_{ki}^2 + q_{ik}\sigma_{ii}^2) \sum_{\ell=1}^{K} h_{\ell k}(x) q_{i\ell}\,.$$

Since $t^{-1} E_x[M_i(t)] \to \infty$ as $t \downarrow 0$, from (9) we see that $(T_t h(x) - h(x))/t$ can stay bounded only if $\sum_{j=1}^{K} h_j(x) q_{ij} = 0$. Assume that this is true. Then using (9) and (18)

$$t^{-1}[T_t h(x) - h(x)] \to \sum_{j=1}^{K} h_j(x) c_j + \sum_{j=1}^{K}\sum_{k=1}^{K} h_{jk}(x)\, \sigma_{jk}^2\,,$$

boundedly pointwise for x on one boundary.

If x lies on more than one boundary, that is, $x_i = 0$ and $x_j = 0$ with $i \neq j$, then $E_x[h(\underset{\sim}{Z}^*(t))] = \hat{o}(t)$ from the first exit distribution for $\underset{\sim}{X}^*$, since h vanishes in every corner.

Now we deal with X on the interior of $\mathbb{R}_+^K$, so that $x_i > 0$ for $1 \leq i \leq K$. We first show pointwise convergence of $(T_t h(x) - h(x))/t$ as $t \downarrow 0$ and then show that this is uniformly bounded on the interior for small enough t. Until $\underset{\sim}{Z}^*$ first hits a boundary it behaves path by path like $\underset{\sim}{X}^*$, so that

$$|E_x\, h(\underset{\sim}{Z}^*(t)) - E_x h(\underset{\sim}{X}^*(t))| = o(t)$$

from the first exit distribution for $\underset{\sim}{X}^*$. Thus

$$(T_t h(x) - h(\mathbf{x}))/t \to \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} \sigma_{ij}^2 \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) + \sum_{k=1}^{K} c_i \frac{\partial f}{\partial x_i}(\mathbf{x}) \text{ as } t \downarrow 0$$

for x on the interior of $\mathbb{R}_+^K$, since the expression on the right is the generator for $\underset{\sim}{X}^*$ on $\mathcal{C}$ (Ito (1961), Section 3.2).

To complete the proof, we show that

$$t^{-1} \sup_{x \in \mathbb{R}_+^K} |T_t h(x) - h(x)| \leq \eta < \infty .$$

We have already shown this for boundary points, and so we only deal with interior points. Using Taylor's theorem,

$$\begin{aligned} &t^{-1}[T_t h(x) - h(x)] \\ &= t^{-1}[\sum_{i=1}^{K} h_i(x) E_x[\Delta Z_i^*(t)] + \sum_{i=1}^{K} \sum_{j=1}^{K} h_{ij}(x) E_x[\Delta Z_i^*(t) \Delta Z_j^*(t)] + \hat{o}(t)], \end{aligned}$$

where the $\hat{o}(t)$ term is due to the uniform bound on the third derivatives of h and bounds on third moments of Brownian motion and third moments of the maximum of Brownian motion. We can bound the second order terms using the calculation above for the generator at a point lying on one boundary. So only the first order terms remain.

We divide the interior of $\mathbb{R}_+^K$ into K+1 (non-disjoint) sets, and deal with each one individually. For $1 \leq k \leq K$, the kth set contains points with $x_k \leq \epsilon/4$. The (K+1)st set contains points with $x_j > \epsilon/4$ for all $j = 1,\ldots,K$. On the (K+1)st set $|E_x(\Delta \underset{\sim}{Z}^*(t)) - E_x(\Delta \underset{\sim}{X}^*(t))| = \hat{o}(t)$ from the first exit distribution for $\underset{\sim}{X}^*$. Since $(\sum_{i=1}^K h_i(x) E_x[\Delta X_i^*(t)])/t$ is known to be bounded, on the (K+1)st set $(\sum_{i=1}^K h_i(x) E_x[\Delta Z_i^*(t)])/t$ is also bounded.

Now consider a point x in the kth set for $1 \leq k \leq K$. If in addition x is in the jth set for $j \neq k$ then $x_j + x_k \leq \epsilon/2$. Then $E_x h(\underset{\sim}{Z}^*(t)) = \hat{o}(t)$ using the first exit distribution for $\underset{\sim}{X}^*$ and the fact that h vanishes in every corner. For these points $(T_t h(x) - h(x))/t$ is therefore bounded. We are left with points in the kth set which are not in any of the others, so that $x_k \leq \epsilon/4$ and $x_j > \epsilon/4$ for $j \neq k$. We have $|E_x(\Delta\underset{\sim}{Z}^*(t)) - E_x(\Delta\underset{\sim}{\bar{Z}}(t))| = \hat{o}(t)$ for these points using the first exit distribution for $\underset{\sim}{X}^*$, where

$$\bar{Z}_\ell(t) = X_\ell^*(t) + q_{k\ell}[M_k(t)]^+ \quad \text{for } 1 \leq \ell \leq K \text{ and } t \geq 0 .$$

In this suffices to show

$$t^{-1} E_x[M_k(t)]^+ \sum_{j=1}^{K} h_j(x) q_{kj} + t^{-1} \sum_{j=1}^{K} h_j(x)\, E_x[X_j^*(t)]$$

is uniformly bounded in x for small enough t. We can immediately dispense with the second term since it is clearly bounded. Using Taylor's theorem,

$$\sum_{j=1}^{K} h_j(x) q_{kj} = \sum_{j=1}^{K} h_j(x^k) q_{kj} + \sup_{x \in \mathbb{R}_+^K} \max_{1 \leq j \leq K} |h_{jk}(x)|x^*$$

with $|x^*| \leq |x|$. Since $\sum_{j=1}^{K} h_j(x^k) q_{kj} = 0$ and $\sup_{x \in \mathbb{R}_+^K} \max_{1 \leq j \leq K} |h_{jk}(x)|$

is finite, we need $(|X|\, E_x[M_k(t)]^+)/t$ bounded in x for small enough t, which follows easily from the first passage and maximum distributions for Brownian motion. This completes the proof.

We now have a justification for calling $\underset{\sim}{Z}^*$ a diffusion with reflection direction $e_i - p_i$. at the boundary corresponding to $Y_i^* = 0$. According to Watanabe (1971), the restrictions on the domain of the generator given by (2) define the angle of reflection. Specifically when a restriction on the domain takes the form that a particular directional derivative at a boundary point must be zero, then the process reflects in this direction at that boundary point.

6.2. The Stationary Distribution

We now derive a partial differential equation which must be satisfied by the density of the stationary distribution of $\underset{\sim}{Z}^*$ (if one exists). Although this equation has not been solved in general, a family of special cases has been solved, and this solution is displayed. In addition, the solution originally proposed by Kobayashi (1974) is substituted into the equations and a family of cases for which his solution works is derived.

We need the following result, a proof of which appears in Azema, Kaplan-Deflo, and Revuz (1966).

Lemma 1. Suppose π is a probability distribution on $\mathbb{R}_+^K$. Then π is a stationary distribution of $\underset{\sim}{Z}^*$ if and only if

$$\int_{\mathbb{R}_+^K} \mathcal{G} h(x)\, \pi(dx) = 0$$

for all $h \in \mathcal{D}$.

We can now prove the following result.

Proposition 3. If $\underset{\sim}{Z}^*$ has a stationary distribution with density π, then π satisfies

$$(19) \qquad \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sigma_{ij}^2 \frac{\partial^2 \pi}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{K} c_i \frac{\partial \pi}{\partial x_i}(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}_+^K$$

and

$$(20) \qquad \frac{1}{2} \sigma_i^2 \frac{\partial \pi}{\partial x_i}(x) + \sum_{j \neq i} \left(\frac{1}{2} \sigma_i^2 p_{ij} + \sigma_{ij}^2\right) \frac{\partial \pi}{\partial x_j}(x) - c_i \pi(x) = 0$$

when $x_i = 0$, $x_j > 0$ for $j \neq i$, for $1 \leq i \leq K$.

Proof. The equations will be obtained by integration by parts on

$$(21) \qquad \int_{\mathbb{R}_+^K} \mathcal{G} h(x)\, \pi(x)\, dx = 0$$

for $h \in \mathcal{D}$. We need to further restrict the class of functions considered to justify the integration by parts. Let

$$I(x) = \{1 \leq i \leq K \mid x_i = 0\} \quad \text{for} \quad x \in \mathbb{R}_+^K$$

and

$$\tilde{\mathbb{R}}_+^K = \{x \in \mathbb{R}_+^K \mid |I(x)| \leq 1\} .$$

Then $\mathbb{R}_+^K$ is simply $\mathbb{R}_+^K$ with all of the corners removed. Let $\mathcal{C}'$ be the set of all functions from $\mathbb{R}_+^K \to \mathbb{R}$ which have derivatives

of all orders when restricted to $\mathbb{R}_+^K$. Since π is a density, Weyl's lemma (McKean (1969), Section 4.2) says that we can assume $\pi \in \mathcal{C}'$. Let $\mathcal{C}''$ be the subset of $\mathcal{C}$ (functions mapping $\mathbb{R}_+^K \to \mathbb{R}$ with bounded derivatives of all orders) which in addition vanish off a compact subset of $\mathbb{R}_+^K$. If we restrict our attention to $h \in \mathcal{C}''$, Fubini's theorem and the fact that $\pi \in \mathcal{C}'$ justify the operations involved in the integrations by parts.

By simple manipulation we get

$$\int_0^\infty h_i(x)\,\pi(x)\,dx_i = -h(x^i)\,g(x^i) - \int_0^\infty h(x)\,\pi_i(x)dx_i \tag{22}$$

for $x \in \mathbb{R}_+^K$ and $1 \le i \le K$,

$$\begin{aligned}(23)\quad &\int_0^\infty\int_0^\infty h_{ij}(x)\,\pi(x)\,dx_i dx_j \\ &= \int_0^\infty h(x^j)\,\pi_i(x^j)dx_i + \int_0^\infty h(x^i)\,\pi_j(x^i)dx_j \\ &\qquad + \int_0^\infty\int_0^\infty h(x)\,\pi_{ij}(x)\,dx_i dx_j\end{aligned}$$

for $x \in \mathbb{R}_+^K$ and $1 \le i \le K$, and

$$\begin{aligned}(24)\quad &\int_{\mathbb{R}_+^{K-1}} dx^i \int_0^\infty h_{ii}(x)\,\pi(x)dx_i \\ &= \int_{\mathbb{R}_+^{K-1}} h_i(x^i)\,\pi(x^i)\,dx^i - \int_{\mathbb{R}_+^{K-1}} h_i(x^i)\,\pi(x^i)dx^i \\ &\qquad + \int_{\mathbb{R}_+^K} h(x)\,\pi_{ii}(x)dx\end{aligned}$$

for $x \in \mathbb{R}_+^K$ and $1 \leq i \leq K$. The last expression still has a term with a derivative on h which we cannot eliminate by an integration by parts because there is no longer an integration with respect to that variable. Since $h \in \mathcal{D} \cap \mathcal{C}$, we know that h satisfies the boundary condition

$$h_i(x^i) = \sum_{j=1}^{K} p_{ij} h_j(x^i) \quad \text{for } x \in \mathbb{R}_+^K \text{ and } 1 \leq i \leq K$$

We can use this to get

$$(25)\quad \int_{\mathbb{R}_+^{K-1}} h_i(x^i)\,\pi(x^i)\,dx^i = \int_{\mathbb{R}_+^{K-1}} \sum_{j=1}^{K} p_{ij} h_j(x^i)\,\pi(x^i)\,dx^i$$
$$= -\sum_{j=1}^{K} \int_{\mathbb{R}_+^{K-1}} p_{ij} h(x^i)\,\pi_j(x^i)\,dx^i.$$

Substituting (22)-(25) into (21),

$$\int_{\mathbb{R}_+^K} h(x)\left[\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}\sigma_{ij}^2 \frac{\partial^2 \pi}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{K} c_i \frac{\partial \pi}{\partial x_i}(x)\right]dx$$
$$+\sum_{i=1}^{K}\int_{\mathbb{R}_+^{K-1}} h(x^i)\left[\frac{1}{2}\sigma_i^2 \frac{\partial \pi}{\partial x_i}(x^i) + \sum_{j\neq i}\left(\frac{1}{2}\sigma_j^2 p_{ji} + \sigma_{ij}^2\right)\frac{\partial \pi}{\partial x_j}(x^i) - c_i \pi(x^i)\right]dx^i = 0.$$

Since this must hold for all $f \in \mathcal{C}''$, all of the terms in square brackets must be zero. This yields (19) and (20).

Proposition 3 only gives a necessary condition for the stationary distribution. Given that a stationary distribution with a density exists, the same condition would be sufficient if it were possible to show uniqueness of the solution for (19) and (20) among the class of

probability density functions. This problem has not been addressed in the theory of partial differential equations and so a sufficient condition is not available.

6.3. Exponential Solutions

The general solution to (19) and (20) is not presently known. However, we are able to solve the equations for a family of special cases. These special cases correspond to limit processes which are achievable by sequence of Jackson networks. We know the explicit form of the stationary distribution for a Jackson network from (1.1). In what follows, we carry through a purely formal calculation to obtain the limit of a sequence of stationary distributions of Jackson networks. Although there is no rigorous result saying that the limit will be the stationary distribution of the limit process, it turns out that the calculated limit solves (19) and (20).

We consider a sequence of Jackson networks. The distributional assumptions yield

$$s_i(n) = [\mu_i(n)]^{-2} \quad \text{and} \quad a_i(n) = [\lambda_i(n)]^{-2}$$

for $1 \leq i \leq K$ and $n \geq 1$. Substituting this into the definition of Σ (equations (3.11) and (3.12)), we obtain

$$\sigma_{ii}^2 = \lim_{n \to \infty} (\lambda_i(n) + \mu_i(n)) = 2\mu_i \tag{26}$$

(27) $$\sigma_{ij}^2 = -(\mu_i p_{ij} + \mu_j p_{ji}),$$

so that

(28) $$\sigma_{ij}^2 = -\frac{1}{2}(\sigma_i^2 p_{ij} + \sigma_j^2 p_{ji}) .$$

This relation determines the family of special cases for which we solve the equations.

From Section 1.3 we know that for a Jackson network with traffic intensity vector $\underset{\sim}{\rho} = (\rho_1, \ldots, \rho_K)$,

$$\pi(n_1, \ldots, n_K) = \prod_{i=1}^{K} (1-\rho_i)\rho_i^{n_i} .$$

Let

$$\Pi(n_1, \ldots, n_K) = \sum_{j_1 \le n_1} \cdots \sum_{j_K \le n_K} \pi(j_1, \ldots, j_K) ,$$

then

(29) $$\Pi(n_1, \ldots, n_K) = \prod_{i=1}^{K} (1 - \rho_i^{n_i+1}) .$$

A simple calculation shows that the condition $(\underset{\sim}{\chi}(n)-\underset{\sim}{\mu}(N))\sqrt{n} \to \underset{\sim}{c}$ as $n \to \infty$ is equivalent to $(\underset{\sim}{\chi}(n) - \underset{\sim}{\mu}(n))\sqrt{n} \to cR$ as $n \to \infty$. This yields $[1 - \rho_i(n)]\sqrt{n} \to (cR)_i/\mu_i$ as $n \to \infty$. We explicitly take limits in (29) to get

$$\Pi(x_1, \ldots, x_K) = \prod_{i=1}^{K} (1-e^{-d_i x_i}) \quad \text{for } x \in \mathbb{R}_+^K$$

where $d_i = -2(cR)_i/\sigma_i^2$. Thus

$$(30) \qquad \pi(x) = \prod_{i=1}^{K} d_i e^{-d_i x_i} ,$$

and substituting this into (19) and (20) verifies that it is a solution.

6.4. Kobayashi's Solution

As was mentioned in Section 1.7, Kobayashi (1974) proposed a diffusion approximation for queueing networks and presented a stationary distribution for his approximation. Although his diffusion process was not completely determined, the stationary distribution he presented actually solves (19) and (20) for a family of special cases. We determine this family of special cases below, showing that it is a subset of the family of cases dealt with in the last section.

Kobayashi's stationary distribution is

$$(31) \qquad \hat{\pi}(x) = \prod_{i=1}^{K} \hat{d}_i e^{-\hat{d}_i x_i} \quad \text{for} \quad x \geq 0 ,$$

where $\hat{\underset{\sim}{d}} = -2\ddagger^{-1}\underset{\sim}{c}$. This always solves (14), but substituting (31) into (20) we determine that for (20) to be satisfied we must have

$$\sum_{j \neq i} \hat{d}_j(\sigma_{ij}^2 + \sigma_i^2 p_{ij}) = 0 \quad \text{for} \quad 1 \leq i \leq K.$$

We take the point of view that subject to $\hat{d}_i > 0$ for $1 \leq i \leq K$, we want the solution to work for any $\hat{d}$. Thus we clearly must have $\sigma_{ij}^2 = -\sigma_i^2 p_{ij}$ for $1 \leq i \leq K$ and $j \neq i$. Since $\sigma_{ij}^2 = \sigma_{ji}^2$ for $1 \leq i, j \leq K$ this implies that $\sigma_{ij}^2 = -\sigma_i^2 p_{ij} = -\sigma_j^2 p_{ji}$ and this is seen to be a subclass of the limits achievable by Jackson networks.

CHAPTER 7

EXTENSIONS

In this chapter we deal with several extensions to the queueing network model originally described in Section 1.2. These extensions were left out of the original model for the sake of improved readability. For some of these extensions, the proof amounts simply to the observation that the old proof still works, while for others the old proof needs a small modification.

7.1. Self-Feedback

The first extension is to certain kinds of non-FIFO disciplines. We will allow any discipline where customers are not preempted in the middle of service, the server is not idle while customers are waiting, and no knowledge about the customers' service time is used in deciding which customer to serve next. The disciplines at different stations can be different. This generality is allowed because it involves no change in the actual vector queue length process. This would not be the case if we were investigating customer waiting times.

We can now show that the assumption $p_{ii} = 0$ for $1 \leq i \leq K$ is without loss of generality. If $p_{ii} > 0$ then it is possible for a customer to return to a station immediately after completing service there. We could define the discipline at stations with $p_{ii} > 0$ to be such that a customer who is routed back to the same station re-enters

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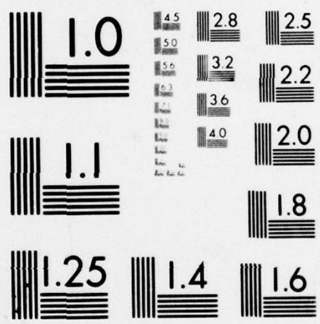

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service immediately. This fits the class of disciplines described above, so the vector queue length process is unchanged. Instead of having the customer leave service and immediately re-enter with probability $p_{ii} > 0$, the same effect is achieved if the service time distribution is redefined to be a geometric mixture of successive convolutions of the original service time distribution. Thus if the service time distribution at station i is $G_i(\cdot)$, the new service time distribution $\hat{G}_i(\cdot)$ is given by

$$\hat{G}_i(t) = \sum_{n=1}^{\infty} (1-p_{ii}) p_{ii}^{n-1} G_i^{*n}(t) \quad \text{for} \quad t \geq 0 ,$$

where G_i^{*n} is the nth convolution of G_i with itself. For the new routing probabilities we obtain

$$\hat{p}_{ii} = 0$$
$$\hat{p}_{ij} = p_{ij}/(1-p_{ii}) \quad \text{for} \quad j \neq i .$$

We have thus transformed the case $p_{ii} > 0$ into $p_{ii} = 0$, without affecting the vector queue length process.

7.2. Multi-Server Stations

The extension to multi-server stations is mathematically straightforward, but notationally difficult. We merely outline the proof since the original proof for single server stations needs very little modification to cover this extension.

We begin by defining the random walk $\underset{\sim}{X}$, for which we need some notation. Let r_k denote the number of servers at station k. Services by server ℓ at station k are distributed as $v_{k\ell}$ and we define

$$\mu_{k\ell} = 1/E[v_{k\ell}] > 0 \quad \text{and} \quad s_{k\ell} = \text{var}(v_{k\ell}) > 0,\ 1 \le i \le K,\ 1 \le \ell \le r_k.$$

Let

$$\mu_k = \sum_{\ell=1}^{r_k} \mu_{k\ell} \quad \text{and} \quad s_k = \mu_k^{-3} \sum_{\ell=1}^{r_k} \mu_{k\ell}^3 s_{k\ell}\ , \quad 1 \le k \le K\ .$$

Let $S_{k\ell} = \{S_{k\ell}(t),\ t \ge 0\}$ be the renewal process formed by potential services of server ℓ at station k. Also, define $S_{k\ell j} = \{S_{k\ell j}(t),\ t \ge 0\}$ to be the counting process for which $S_{k\ell j}(t)$ is the number of potential services completed by server ℓ at station k during $[0,t]$ where routing indicators pointed to station j. Now define the random walk $\underset{\sim}{X} = \{\underset{\sim}{X}(t),\ t \ge 0\}$ by

$$X_k(t) = A_k(t) - \sum_{\ell=1}^{r_k} S_{k\ell}(t) + \sum_{j=1}^{K} \sum_{\ell=1}^{r_j} S_{j\ell k}(t),\quad 1 \le k \le K,\ t \ge 0\ .$$

We assume that all of the processes A_k, $S_{k\ell}$ for $1 \le k, \ell \le K$ are mutually independent so that the sequence of random walks constructed as in the single server case converges to a multidimensional Brownian motion. If we define

$$\underset{\sim}{X}^{*n}(t) = n^{-1/2}\underset{\sim}{X}^n(nt) \quad \text{for} \quad 0 \le t \le 1,\ n \ge 1\ ,$$

then in complete analogy with Theorem 1 we obtain $\underset{\sim}{X}^{*n} \Rightarrow \underset{\sim}{X}^{*}$ as $n \to \infty$, where X^* is exactly as defined in Theorem 1.

The rest of the convergence proof proceeds as before. The Borovkov network is changed to correspond to Borovkov's modification for multi-server queues. The main change from the single server case is that a customer is assigned to the server who would end up completing his service first. This is not always the first server to become idle. Otherwise the Borovkov network is the same as before. The I/O network is defined as before, and the RRW is constructed from $\underset{\sim}{X}$ exactly as before. The proofs of Propositions 5.1, 5.2, and 5.3 are the same as before, with the only added complexity being a more cumbersome notation. The final result is that Theorem 3 holds for the multi-server case.

7.3. Non-Renewal Input

In the original model it was assumed that the input processes A_k are all renewal processes. The only place this assumption was actually used was in the proof of Theorem 1. The renewal nature of A_k was used to show that the scaled and normalized processes A_k^{*n} converge weakly to Brownian motion. Instead of assuming A_k to be a renewal process, we can simply assume that the sequence $A_k^n(n\cdot)$ can be scaled and normalized to form a sequence A_k^{*n} which converges weakly to Brownian motion. The scaling parameters play the role of arrival rates.

To show that this generalization is actually useful, we give an example where it can be applied. The example involves an arrival process introduced by Anderson (1975) which he calls an appointments process. Arrivals are scheduled for times $i \geq 1$, with the ith scheduled arrival being unpunctual by an amount w_i, where $\{w_i, i \geq 1\}$ forms an i.i.d. sequence having common distribution $F(\cdot)$. Thus the ith scheduled arrival actually arrives at $i + w_i$. We assume F to be an imperfect distribution function and let $q = 1 - F(\infty) > 0$. This corresponds to having a probability $q > 0$ that a scheduled customer will never arrive. It is also assumed that $\int_{-\infty}^{\infty} xF(dx) < \infty$. Let $A = \{A(t), t \geq 0\}$ be the counting process where $A(t)$ is the number of arrivals during $(-\infty, t]$ and set

$$A^{*n}(t) = [q(1-q)n]^{-1/2} \, [A(nt) - \sum_{i=0}^{\infty} F(nt-i)], \quad 0 \leq t \leq 1, \; n \geq 1.$$

Anderson's main result is that $A^{*n} \Rightarrow \beta$ where β is a standard Brownian motion. Thus we can use an appointment process as input for any number of stations in the network and obtain the same diffusion limit as in the original model. In order to match the variance here with the original model, set $a_i = q_i(1-q_i)$ if station i has an appointments process input.

There are other cases where the additional generality allowed by this section is helpful. An important case is that of thinned and superimposed renewal processes. Thus a station could have two or more

arrival sources, or two stations could share an arrival source. These possibilities increase the number of situations where the queueing network model treated here can be used.

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Abstract Report No. 76 Author: Martin Ira Reiman

The principle purpose of this report is to state and prove a limit rem which justifies a diffusion approximation for general queueing orks.

~~The~~ structure of the network is as follows. There are K single er stations, each with independent and identically distributed (i.i.d) ice times. Each station may have its own input from outside the network, interarrival times being iid. In addition, all of the service and rarrival time sequences are mutually independent. The service and rarrival distributions can be arbitrary as long as they have finite ances. Customers completing service at a station are routed randomly er to another station or out of the network. The routing probabilities elements of a K x K substochastic matrix P, called the routing ix. Each row defect gives the probability of a customer leaving the em after completing service at the associated station. The routing cators are independent of all other events in the process.

We are interested in the K-dimensional vector queue length process is investigated the network. Because of the general form assumed for the interarrival service distributions, the process has no special structure such as the ov property. In this generality, the network has proven to be actable, hence the desire for an approximation.

In direct analogy with standard single server queueing systems, it is ible to define a traffic intensity for each station in the network. y traffic is said to hold when all stations have traffic intensities e to unity. Mathematically, heavy traffic is interpreted through ideration of a sequence of queueing networks indexed (say) by n, each its own parameters, defined in such a way that the traffic intensity ach station approaches unity as n approaches infinity. The vector e length process tends to grow without bound when the traffic nsities are greater than unity or approach it from below, so it is ssary to normalize the sequence of processes to obtain a nondegenerate t. Thus the main result states that if we have a properly normalized ence of queueing networks with traffic intensities approaching unity the correct rate), it converges to a certain multi-dimensional diffusion ess.

We next turn to a study of the limit diffusion process. The state space he limit process is the K-dimensional non-negative orthant. On the rior of its state space the process behaves as a multidimensional nian motion with an easily computed drift vector and covariance matrix. ach boundary surface the process reflects instantaneously. The ctions of reflection are given by a simple expression involving only routing matrix. After proving that the limit process is a diffusion, ~~ompute~~ its generator is computed, justifying the above description.

We then derive a necessary condition for a probability density function e a stationary density for the limit process. Specifically, we show that density of a stationary distribution, if one exists, must satisfy a ain partial differential equation with boundary conditions. Although equation has not been solved in general, a family of special cases is ed.


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